

On k -generalized Lucas sequence with its triangle

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Abstract: In this paper, we investigate several identities of k -generalized Lucas numbers with k -generalized Fibonacci numbers. We also establish a link between generalized s -Lucas triangle and bi ^{s} nomial coefficients given by the coefficients of the development of a power of $(1 + x + x^2 + \cdots + x^s)$, with $s \in \mathbb{N}$.

Key words: k -generalized Lucas sequence, arithmetic triangle, recurrence relation, bi ^{s} nomial coefficient

1. Introduction

Let $\{G_n\}$ be a sequence defined by second-order linear recurrence relation $G_n = AG_{n-1} + BG_{n-2}$, $n \geq 2$ where A , B , G_0 and G_1 are given numbers. Assume that the sequence $\{H_n\}$ is defined by the same recurrence relation of $\{G_n\}$ with $H_0 = 2G_1 - AG_0$ and $H_1 = AG_1 + 2BG_0$. $\{H_n\}$ is called the associate sequence of $\{G_n\}$ (see [18]). Table 1 presents several well-known sequences with their associate sequences and A-numbers in Sloane's Encyclopedia of Integer Sequences[†].

Table 1. second-order well-known sequences

A	B	G_0	G_1	H_0	H_1	Sequence	Associate sequence	A-numbers
1	1	0	1	2	1	Fibonacci	Lucas	A000045, A000032
2	1	0	1	2	2	Pell	Pell-Lucas	A000129, A002203
1	2	0	1	2	1	Jacobsthal	Jacobsthal-Lucas	A001045, A014551
6	-1	0	1	2	6	Balancing	Balancing-Lucas	A001109, A003499

There are several generalizations of the Fibonacci sequence. One of the generalizations relating to order is k -generalized Fibonacci sequence. For $k \geq 2$, k -generalized Fibonacci sequence $\{F_n^{(k)}\}$ is defined by the

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[†]Sloane NJA, The Online Encyclopedia of Integer Sequences. Available online at <http://oeis.org/>.

following recurrence relation

$$F_n^{(k)} = \begin{cases} 0, & \text{if } n = -1, \dots, -k+1; \\ 1, & \text{if } n = 0; \\ F_{n-1}^{(k)} + F_{n-2}^{(k)} + \dots + F_{n-k}^{(k)} & \text{if } n > 1. \end{cases}$$

For the few values of k , we give in Table 2 of these numbers containing the A-number, according to the On-Line Encyclopedia of Integer Sequences (OEIS)[‡].

Table 2. k -order well-known sequences

k	Sequence name	Terms of the sequence	A – numbers
2	Fibonacci	1, 1, 2, 3, 5, 8, 13, ...	A000045
3	Tribonacci	1, 1, 2, 4, 7, 13, 24, ...	A000073
4	Tetranacci	1, 2, 4, 8, 15, 29, 56, ...	A000078
5	Pentanacci	1, 1, 2, 4, 8, 16, 31, ...	A001591

The Binet form of the k -generalized Fibonacci sequence is given by Dresden and Du [13] as follows

Theorem 1.1 For $F_n^{(k)}$ the n^{th} k -generalized Fibonacci number, then

$$F_n^{(k)} = \sum_{i=1}^k \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}$$

for $\alpha_1, \alpha_2, \dots, \alpha_k$ the roots of $x^k - x^{k-1} - \dots - 1 = 0$.

There are also many other ways to represent the terms of k -generalized Fibonacci numbers (see [16], [14], [15], [17]).

By the motivation of the definition "associate sequence", we give the definition of the associate sequence of $\{F_n^{(k)}\}_n$ which we call it as k -order Lucas sequence $\{L_n^{(k)}\}_n$.

Definition 1.2 Let $k \geq 2$ is an integer. The k -generalized Lucas sequence $\{L_n^{(k)}\}_n$ by the following recurrence relation

$$L_n^{(k)} = L_{n-1}^{(k)} + L_{n-2}^{(k)} + \dots + L_{n-k}^{(k)}, \quad (n \geq -k+2) \quad (1.1)$$

with the initials $L_0^{(k)} = k$, $L_1^{(k)} = 1$, $L_2^{(k)} = 3$, ..., $L_{k-1}^{(k)} = 2^{k-1} - 1$.

The explicit formulas of the k -generalized Fibonacci and Lucas sequences are given by Belbachir and Bencherif [7] as follows:

$$F_n^{(k)} = \sum_{j_1+2j_2+\dots+kj_k=n} \binom{j_1+j_2+\dots+j_k}{j_1, j_2, \dots, j_k}$$

[‡]Sloane NJA, The Online Encyclopedia of Integer Sequences. Available online at <http://oeis.org/>.

and

$$L_n^{(k)} = \lambda_0 y_{n-k+1} + \lambda_1 y_{n-k+2} + \cdots + \lambda_{k-1} y_n,$$

with $\lambda_j = -\sum_{i=j}^{k-j} L_i^{(k)}$ for $0 \leq j \leq k-j$, and $y_n = \sum_{j_1+2j_2+\cdots+kj_k=n} \binom{j_1+j_2+\cdots+j_k}{j_1, j_2, \dots, j_k}$ for $n > -k$.

We note that the case $k = 2$ gives Lucas numbers and the case $k = 3$ gives Tribonacci-Lucas numbers (see [20]). There are two parts in the present paper; the first one gives combinatorial identities for k -generalized Lucas numbers and extends identities between Fibonacci and Lucas numbers. In the second part, we give several relations between k -generalized Lucas numbers and binomial coefficients.

2. Connections with k -generalized Fibonacci and Lucas numbers

Before giving our results of this section, we recall that the Binet formula of k -generalized Lucas numbers is given by the following result

Lemma 2.1 *Let $\alpha_1, \alpha_2, \dots, \alpha_k$ are the distinct roots of $x^k - x^{k-1} - \cdots - 1 = 0$. Then, we have*

$$L_n^{(k)} = \alpha_1^n + \alpha_2^n + \cdots + \alpha_k^n.$$

This result is well-known. We give the proof for convenience.

Proof It is known that the term $L_n^{(k)}$ can be written by

$$L_n^{(k)} = A_1 \alpha_1^n + A_2 \alpha_2^n + \cdots + A_k \alpha_k^n$$

where A_i are real numbers. Our aim to find the numbers A_i for $i = 1, 2, \dots, k$. To find these values, we get the following system of equations

$$\begin{aligned} L_0^{(k)} &= A_1 + A_2 + \cdots + A_k \\ L_1^{(k)} &= A_1 \alpha_1 + A_2 \alpha_2 + \cdots + A_k \alpha_k \\ L_2^{(k)} &= A_1 \alpha_1^2 + A_2 \alpha_2^2 + \cdots + A_k \alpha_k^2 \\ &\vdots \\ L_{n-1}^{(k)} &= A_1 \alpha_1^{n-1} + A_2 \alpha_2^{n-1} + \cdots + A_k \alpha_k^{n-1}. \end{aligned}$$

By using the Cramer's rule, we obtain $A_1 = A_2 = \cdots = A_k = 1$. □

From now on, we generalize several well-known identities between Fibonacci and Lucas numbers. To prove these identities, we will use the Binet type formulas for k -generalized Fibonacci and Lucas numbers. We know that the following identity is given by Ramirez and Sirvent [19]. Here, we give its proof by using Binet formulas.

Theorem 2.2 *Let k and n nonnegative integers with $k \geq 2$, then we have*

$$\sum_{i=1}^k i F_{n-i+1}^{(k)} = L_n^{(k)}. \quad (2.1)$$

Proof By using the Binet Formula of k -generalized Fibonacci numbers, we get

$$\begin{aligned}
 \sum_{i=1}^k iF_{n-i+1}^{(k)} &= F_n^{(k)} + 2F_{n-1}^{(k)} + 3F_{n-2}^{(k)} + \cdots + kF_{n-k+1}^{(k)} \\
 &= \sum_{i=1}^k \frac{(\alpha_i - 1)\alpha_i^{n-1}}{2 + (k+1)(\alpha_i - 2)} + 2 \frac{(\alpha_i - 1)\alpha_i^{n-2}}{2 + (k+1)(\alpha_i - 2)} + \cdots + k \frac{(\alpha_i - 1)\alpha_i^{n-k}}{2 + (k+1)(\alpha_i - 2)} \\
 &= \sum_{i=1}^k \frac{(\alpha_i - 1)\alpha_i^{n-k} [k + (k-1)\alpha_i + (k-2)\alpha_i^2 + \cdots + 2\alpha_i^{k-2} + \alpha_i^{k-1}]}{2 + (k+1)(\alpha_i - 2)} \\
 &= \sum_{i=1}^k \frac{\alpha_i^n (\alpha_i^k + \alpha_i^{k-1} + \alpha_i^{k-2} + \cdots + \alpha_i - k)}{\alpha_i^k [(k+1)\alpha_i - 2k]}
 \end{aligned}$$

After using the facts $\alpha_i^k = \alpha_i^{k-1} + \alpha_i^{k-2} + \cdots + 1$ and $\alpha_i^k + \alpha_i^{k-1} + \alpha_i^{k-2} + \cdots + \alpha_i - k = 2\alpha_i^k - 1 - k$, we have

$$\begin{aligned}
 &\sum_{i=1}^k \frac{\alpha_i^n (\alpha_i^k + \alpha_i^{k-1} + \alpha_i^{k-2} + \cdots + \alpha_i - k)}{\alpha_i^k [(k+1)\alpha_i - 2k]} \\
 &= \sum_{i=1}^k \frac{\alpha_i^n (2\alpha_i^k - 1 - k)}{(k+1)\alpha_i^k + (1-k)\alpha_i^k - 1 - k} \\
 &= \sum_{i=1}^k \alpha_i^n = L_n^{(k)}.
 \end{aligned}$$

For $k = 2$,

$$F_n + 2F_{n-1} = L_n.$$

□

We have also the following identity.

Theorem 2.3 Let k and n be nonnegative integers with $k \geq 2$, then we have

$$L_{n-1}^{(k)} + L_{n+1}^{(k)} = 2kF_n^{(k)} + \sum_{i=0}^{k-2} (3 - k + 2i)F_{n-i}^{(k)}. \quad (2.2)$$

Proof Together with the Binet Formula for k -generalized Fibonacci number, we get the followings

$$\begin{aligned}
& 2kF_n^{(k)} + \sum_{i=0}^{k-2} (3-k+2i)F_{n-i}^{(k)} \\
&= (k+3)F_n^{(k)} + (5-k)F_{n-1}^{(k)} + (7-k)F_{n-2}^{(k)} + \cdots + (k-3)F_{n-k+3}^{(k)} + (k-1)F_{n-k+2}^{(k)} \\
&= \sum_{i=1}^k \frac{(k+3)(\alpha_i-1)\alpha_i^{n-1}}{2+(k+1)(\alpha_i-2)} + \frac{(5-k)(\alpha_i-1)\alpha_i^{n-2}}{2+(k+1)(\alpha_i-2)} + \frac{(7-k)(\alpha_i-1)\alpha_i^{n-3}}{2+(k+1)(\alpha_i-2)} + \\
&\quad \cdots + \frac{(k-3)(\alpha_i-1)\alpha_i^{n-k+2}}{2+(k+1)(\alpha_i-2)} + \frac{(k-1)(\alpha_i-1)\alpha_i^{n-k+1}}{2+(k+1)(\alpha_i-2)} \\
&= \sum_{i=1}^k \frac{(k+3)\alpha_i^n + (2-2k)\alpha_i^{n-1} + 2\alpha_i^{n-2} + 2\alpha_i^{n-3} + \cdots + 2\alpha_i^{n-k+2} + (1-k)\alpha_i^{n-k+1}}{(k+1)\alpha_i - 2k} \\
&= \sum_{i=1}^k \frac{\alpha_i^{n-k+1}(\alpha_i^{k-1} + 2\alpha_i^{k-2} + \alpha_i^{k-3} + \alpha_i^{k-4} + \cdots + \alpha_i + 1)[(k+1)\alpha_i - 2k]}{(k+1)\alpha_i - 2k} \\
&= \sum_{i=1}^k \alpha_i^{n-k+1}(\alpha_i^{k-1} + 2\alpha_i^{k-2} + \alpha_i^{k-3} + \alpha_i^{k-4} + \cdots + \alpha_i + 1) \\
&= \sum_{i=1}^k \alpha_i^{n+1} + \alpha_i^{n-1} = L_{n+1}^{(k)} + L_{n-1}^{(k)}.
\end{aligned}$$

□

This generalizes the identity

$$L_{n-1} + L_{n-1} = 5F_n.$$

Since one can prove the following theorem as before, we do not give the proof.

Theorem 2.4 Assume that k and n are nonnegative integers, with $k \geq 2$, we have

$$L_{n+k-2}^{(k)} = kF_{n+k-1}^{(k)} - \sum_{i=1}^{k-1} iF_{n+i-1}^{(k)}, \quad (2.3)$$

$$L_{n-2}^{(k)} = (2k-1)F_{n-1}^{(k)} - F_{n+k-2}^{(k)} + \sum_{i=1}^{k-3} (k-i-2)F_{n+i-1}^{(k)}. \quad (2.4)$$

These generalize the identities $L_n = 2F_n - F_{n-1}$ and $L_{n-2} = 3F_{n-1} - F_n$.

3. The generalized s -Lucas triangle

In this section, we propose a generalization of Lucas and Tribonacci-Lucas triangles, such that the sum of elements located along the direction $(1, 1)$ (see [8] for the details about the notion of direction) in the generalized s -Lucas triangle gives the terms of $(s+1)$ -generalized Lucas sequence, the explicit formula is given. We establish

a link between generalized s -Lucas triangle and bi s nomial coefficients. We also give the recurrence relation for the sum of elements lying over the finite direction of the generalized s -Lucas triangle.

Alladi and Hoggat [1] have defined the Tribonacci triangle, (this triangle is a generalization of Pascal triangle) and proved that the sum of elements lying over the principal diagonal rays in the Tribonacci triangle gives the Tribonacci sequence

$$T_{n+1} = T_n + T_{n-1} + T_{n-2},$$

with $T_0 = 0, T_1 = 1, T_2 = 1$.

Denote by $\binom{n}{k}_{[2]}$ the element in the n^{th} row and k^{th} column of the Tribonacci triangle, the triangle is produced by the recurrence relation,

$$\binom{n}{k}_{[2]} = \binom{n-1}{k}_{[2]} + \binom{n-1}{k-1}_{[2]} + \binom{n-2}{k-1}_{[2]},$$

where $\binom{n}{0}_{[2]} = \binom{n}{n}_{[2]} = 1$. We use the convention $\binom{n}{k}_{[2]} = 0$ for $k \notin \{0, \dots, n\}$. We present several values of $\binom{n}{k}_{[2]}$ in Table 3.

Table 3. Tribonacci triangle.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	3	1							
3	1	5	5	1						
4	1	7	13	7	1					
5	1	9	25	25	9	1				
6	1	11	41	63	41	11	1			
7	1	13	61	129	129	61	13	1		
8	1	15	85	231	321	231	85	15	1	
9	1	17	113	377	681	681	377	113	17	1

Moreover, Barry [6] has shown that for $0 \leq k \leq n$ these coefficients satisfy the relation

$$\binom{n}{k}_{[2]} = \sum_{j=0}^k \binom{k}{j} \binom{n-j}{k}, \quad (3.1)$$

we recall that the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and we use the convention $\binom{n}{k} = 0$ for $k > n$, $k < 0$ or $n < 0$.

Recently, Yilmaz and Taskara [20] have defined the Tribonacci-Lucas triangle which is a generalization of Lucas triangle and they have shown that the sum of elements lying over the principal diagonal rays in this triangle gives the Tribonacci-Lucas sequence.

$$K_n = K_{n-1} + K_{n-2} + K_{n-3},$$

with $K_0 = 3, K_1 = 1, K_2 = 3$.

Denote by $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{[2]}$ the element in the n^{th} row and k^{th} column of the Tribonacci-Lucas triangle, the triangle is

produced by the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[2]} = \begin{bmatrix} n-1 \\ k \end{bmatrix}_{[2]} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{[2]} + \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{[2]},$$

where $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{[2]} = 3$, $\begin{bmatrix} n \\ 0 \end{bmatrix}_{[2]} = 1$ and $\begin{bmatrix} n \\ n \end{bmatrix}_{[2]} = 2$ for $n \geq 1$. We use the convention $\begin{bmatrix} n \\ k \end{bmatrix}_{[2]} = 0$ for $k \notin \{0, \dots, n\}$.

Table 4 shows the values of $\begin{bmatrix} n \\ k \end{bmatrix}_{[2]}$ for special cases k and n .

Table 4. Tribonacci-Lucas triangle.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	3									
1	1	2								
2	1	6	2							
3	1	8	10	2						
4	1	10	24	14	2					
5	1	12	42	48	18	2				
6	1	14	64	114	80	22	2			
7	1	16	90	220	242	120	26	2		
8	1	18	120	374	576	442	168	30	2	
9	1	20	154	584	1170	1260	730	224	34	2

The explicit formula of the coefficients of the Tribonacci-Lucas triangle is given by, see [20], for $n \geq 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[2]} = \sum_{j=0}^k \binom{k}{j} \binom{n-j}{k} \frac{n+k}{n-j}.$$

3.1. The s -Pascal triangle

The bi s nomial coefficient $\binom{n}{k}_s$ is the element in the n^{th} row and k^{th} column of s -Pascal triangle. The s -Pascal triangle is constructed by the following recurrence relation, see [3, 5, 10],

$$\binom{n}{k}_s = \binom{n-1}{k}_s + \binom{n-1}{k-1}_s + \dots + \binom{n-1}{k-s}_s. \quad (3.2)$$

Using the classical binomial coefficient, one has

$$\binom{n}{k}_s = \sum_{j_1+j_2+\dots+j_s=k} \binom{n}{j_1} \binom{j_1}{j_2} \dots \binom{j_{s-1}}{j_s}. \quad (3.3)$$

Some other readily well known established properties are:

the symmetry relation

$$\binom{n}{k}_s = \binom{n}{sn-k}_s, \quad (3.4)$$

the diagonal recurrence relation

$$\binom{n}{k}_s = \sum_{j=0}^n \binom{n}{j} \binom{j}{k-j}_{s-1}, \quad (3.5)$$

and de Moivre's expression (see [11, 12])

$$\binom{n}{k}_s = \sum_{j=0}^n (-1)^j \binom{n}{j} \binom{k-j(s+1)+n-1}{n-1}. \quad (3.6)$$

For $s = 2$, we have bitrinomial triangle illustrated in Table 5, see Sloane as A027907[§].

Table 5. Bitrinomial triangle ($s = 2$).

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	1											
2	1	2	3	2	1								
3	1	3	6	7	6	3	1						
4	1	4	10	16	19	16	10	4	1				
5	1	5	15	30	45	51	45	30	15	5	1		
6	1	6	21	50	90	126	141	126	90	50	21	6	1

4. Quasi s -Pascal triangle

Recently, Amrouche and Belbachir [2–4] have defined a generalization of Pascal and Delannoy triangles, called quasi s -Pascal triangle. They denoted by $\binom{n}{k}_{[s]}$ the coefficient in the n^{th} row and k^{th} column of this triangle such that the coefficient $\binom{n}{k}_{[s]}$ satisfies,

$$\binom{n}{k}_{[s]} = \binom{n-1}{k}_{[s]} + \binom{n-1}{k-1}_{[s]} + \binom{n-2}{k-1}_{[s]} + \cdots + \binom{n-s}{k-1}_{[s]}. \quad (4.1)$$

The following result gives the explicit formula of the coefficients of the quasi s -Pascal triangle in terms of binomial coefficients.

Theorem 4.1 [3] *The quasi-binomial coefficient $\binom{n}{k}_{[s]}$ satisfies*

$$\binom{n}{k}_{[s]} = \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n - \sum_{i=1}^{s-1} j_i}{k}. \quad (4.2)$$

The sum of elements located along the direction $(1, 1)$ in the quasi s -Pascal triangle gives the terms of $(s+1)$ -generalized Fibonacci sequence.

Let $(T_{n,s})_n$ be the terms of the $(s+1)$ -generalized Fibonacci sequence, for $n \geq 0$

$$T_{n+1,s} := \sum_k \binom{n-k}{k}_{[s]},$$

[§]Sloane NJA, The Online Encyclopedia of Integer Sequences. Available online at <http://oeis.org/>.

with $T_{0,s} = 0$.

Theorem 4.2 [3] For $n \geq 0$, $(T_{n,s})_n$ satisfies the following recurrence relation

$$T_{n+1,s} = T_{n,s} + T_{n-1,s} + \cdots + T_{n-s,s},$$

with $T_{1,s} = 1, T_{i,s} = 0$ for $i \in \{0, -1, \dots, -(s-1)\}$.

Amrouche and Belbachir [3] have extended the last result, they considered the sum of elements located along the finite direction (α, r) ($r + \alpha > 0$, $r \in \mathbb{N}$, $0 \leq p < r$ and $\alpha \in \mathbb{Z}$) in the quasi s -Pascal triangle, (for the details about the direction in arithmetic triangles one can see [2, 4, 8, 9]).

Let $T_{n,s}^{(\alpha,\beta,r)}$ be the terms of the sequence obtained by this sum, for $n \geq 0$

$$T_{n+1,s}^{(\alpha,\beta,r)} := \sum_k \binom{n-rk}{\beta+\alpha k}_{[s]}, \text{ with } T_{0,s}^{(\alpha,\beta,r)} = 0.$$

Theorem 4.3 [3] For $n \geq \alpha s + r$, $(T_{n,s}^{(\alpha,\beta,r)})_n$ satisfies the following linear recurrence relation

$$\sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} T_{n-i,s}^{(\alpha,\beta,r)} = \sum_{i=0}^{\alpha(s-1)} \binom{\alpha}{i}_{s-1} T_{n-\alpha-r-i,s}^{(\alpha,\beta,r)}. \quad (4.3)$$

5. The quasi s -Lucas triangle

We propose an extension of Lucas and Tribonacci-Lucas triangle called quasi s -Lucas triangle such that the sum of elements located along the direction $(1, 1)$ gives the terms of $(s+1)$ -generalized Lucas sequence.

Definition 5.1 Let $\begin{bmatrix} n \\ k \end{bmatrix}_{[s]}$ the element of the n^{th} line and k^{th} column of s -generalized Lucas triangle

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} = \binom{n}{k}_{[s]} + \binom{n-1}{k-1}_{[s]} + 2 \binom{n-2}{k-1}_{[s]} + \cdots + s \binom{n-s}{k-1}_{[s]}, \quad (5.1)$$

with $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{[s]} = s+1$.

Theorem 5.2

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} = \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n - \sum_{i=1}^{s-1} j_i}{k} \frac{n+k}{n-j_1-\cdots-j_{s-1}}.$$

Proof By the relation (5.1) and (4.2), we have

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} &= \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k} \\
&+ \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k-1}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-1}{k-1} \\
&+ 2 \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k-1}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-2}{k-1} \\
&\vdots \\
&+ (s-1) \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k-1}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-(s-1)}{k-1} \\
&+ s \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k-1}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-s}{k-1} \\
&= \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-2}} \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k} \\
&+ \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-2}} \sum_{j_{s-1}} \binom{k-1}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-1}{k-1} \\
&+ 2 \sum_{j'_1} \sum_{j_2} \cdots \sum_{j_{s-2}} \sum_{j_{s-1}} \binom{k-1}{j'_1-1} \binom{j'_1-1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j'_1-\cdots-j_{s-2}-j_{s-1}-1}{k-1} \\
&\vdots \\
&+ (s-1) \sum_{j'_1} \sum_{j'_2} \cdots \sum_{j'_{s-2}} \sum_{j_{s-1}} \binom{k-1}{j'_1-1} \binom{j'_1-1}{j'_2-1} \cdots \binom{j'_{s-2}-1}{j_{s-1}} \binom{n-j'_1-\cdots-j'_{s-2}-j_{s-1}-1}{k-1} \\
&+ s \sum_{j'_1} \sum_{j'_2} \cdots \sum_{j'_{s-2}} \sum_{j'_{s-1}} \binom{k-1}{j'_1-1} \binom{j'_1-1}{j'_2-1} \cdots \binom{j'_{s-2}-1}{j'_{s-1}-1} \binom{n-j'_1-\cdots-j'_{s-2}-j'_{s-1}-1}{k-1}, \\
&\text{as } \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \text{ then} \\
&\sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-2}} \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-1}{k-1} \\
&= \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k-1}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-1}{k-1}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j'_1} \sum_{j'_2} \cdots \sum_{j'_{s-2}} \sum_{j'_{s-1}} \binom{k-1}{j'_1-1} \binom{j'_1-1}{j'_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j'_1-\cdots-j_{s-2}-j_{s-1}-1}{k-1} \\
& \vdots \\
& + \sum_{j'_1} \sum_{j'_2} \cdots \sum_{j'_{s-2}} \sum_{j'_{s-1}} \binom{k-1}{j'_1-1} \binom{j'_1-1}{j'_2-1} \cdots \binom{j'_{s-2}-1}{j_{s-1}} \binom{n-j'_1-\cdots-j'_{s-2}-j_{s-1}-1}{k-1} \\
& + \sum_{j'_1} \sum_{j'_2} \cdots \sum_{j'_{s-2}} \sum_{j'_{s-1}} \binom{k-1}{j'_1-1} \binom{j'_1-1}{j'_2-1} \cdots \binom{j'_{s-2}-1}{j'_{s-1}-1} \binom{n-j'_1-\cdots-j'_{s-2}-j'_{s-1}-1}{k-1},
\end{aligned}$$

then we obtain

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} &= \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-2}} \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k} \\
& + \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-2}} \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-1}{k-1} \\
& + \sum_{j'_1} \sum_{j'_2} \cdots \sum_{j'_{s-2}} \sum_{j'_{s-1}} \binom{k-1}{j'_1-1} \binom{j'_1-1}{j'_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j'_1-\cdots-j_{s-2}-j_{s-1}-1}{k-1} \\
& \vdots \\
& + (s-2) \sum_{j'_1} \sum_{j'_2} \cdots \sum_{j'_{s-2}} \sum_{j'_{s-1}} \binom{k-1}{j'_1-1} \binom{j'_1-1}{j'_2-1} \cdots \binom{j'_{s-2}-1}{j_{s-1}} \binom{n-j'_1-\cdots-j'_{s-2}-j_{s-1}-1}{k-1} \\
& + (s-1) \sum_{j'_1} \sum_{j'_2} \cdots \sum_{j'_{s-2}} \sum_{j'_{s-1}} \binom{k-1}{j'_1-1} \binom{j'_1-1}{j'_2-1} \cdots \binom{j'_{s-2}-1}{j'_{s-1}-1} \binom{n-j'_1-\cdots-j'_{s-2}-j'_{s-1}-1}{k-1} \\
& = \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k} \\
& + \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k} \frac{k}{n-j_1-\cdots-j_{s-2}-j_{s-1}} \\
& + \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k} \frac{j_1-j_2}{n-j_1-\cdots-j_{s-2}-j_{s-1}} \\
& \vdots \\
& + \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k} \frac{(s-2)(j_{s-2}-j_{s-1})}{n-j_1-\cdots-j_{s-2}-j_{s-1}}
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k} \frac{(s-1)j_{s-1}}{n-j_1-\cdots-j_{s-2}-j_{s-1}} \\
 & = \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n-\sum_{i=1}^{s-1} j_i}{k} \frac{n+k}{n-j_1-\cdots-j_{s-1}}.
 \end{aligned}$$

□

For $s = 1$ and $s = 2$, we obtain the Lucas and Tribonacci-Lucas triangles respectively.

The sum of elements located along the direction $(1, 1)$ in the generalized s -Lucas triangle gives the $(s + 1)$ -generalized Lucas sequence $\{L_n^{(s)}\}_{n \geq 0}$.

Let the sequence $\{L_n^{(s)}\}_{n \geq 0}$ given by

$$L_n^{(s+1)} = \sum_k \left[\begin{matrix} n-k \\ k \end{matrix} \right]_{[s]}. \quad (5.2)$$

We present the following theorem in Theorem 2.2. Here, we give another proof of the theorem by using (5.2).

Theorem 5.3 *The sequence $\{L_n^{(s)}\}_{n \geq 0}$ satisfies the following recurrence relation*

$$L_n^{(s+1)} = F_{n+1}^{(s+1)} + F_{n-1}^{(s+1)} + 2F_{n-2}^{(s+1)} + \cdots + sF_{n-s}^{(s+1)}. \quad (5.3)$$

Proof From (5.1) and (5.2) we have

$$\begin{aligned}
 L_n^{(s+1)} &= \sum_k \binom{n-k}{k}_{[s]} + \sum_k \binom{n-k-1}{k-1}_{[s]} + 2 \sum_k \binom{n-k-2}{k-1}_{[s]} + \cdots \\
 &\quad \cdots + s \sum_k \binom{n-k-s}{k-1}_{[s]} \\
 &= \sum_k \binom{n-k}{k}_{[s]} + \sum_{k'} \binom{n-k'-2}{k'}_{[s]} + 2 \sum_{k'} \binom{n-k'-3}{k'}_{[s]} + \cdots \\
 &\quad \cdots + s \sum_{k'} \binom{n-k'-s-1}{k'}_{[s]} \\
 &= F_{n+1}^{(s+1)} + F_{n-1}^{(s+1)} + 2F_{n-2}^{(s+1)} + \cdots + F_{n-s}^{(s+1)}.
 \end{aligned}$$

□

5.1. Link between generalized s -Lucas triangle and bi^snomial coefficients

The following result establishes the relation between the generalized s -Lucas triangle and s -Pascal triangle.

Theorem 5.4 *For fixed nonnegative integers n, k and s , we have*

$$\left[\begin{matrix} n \\ k \end{matrix} \right]_{[s]} = \sum_i \binom{n-i}{k} \binom{k}{i}_{s-1} \frac{n+k}{n-i}.$$

Proof We have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} = \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n - \sum_{i=1}^{s-1} j_i}{k} \frac{n+k}{n-j_1-\cdots-j_{s-1}}.$$

Considering the summations by blocks $j_1 + j_2 + \cdots + j_{s-1} = i$, we get

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} = \sum_i \binom{n-i}{k} \frac{n+k}{n-i} \sum_{j_1+j_2+\cdots+j_{s-1}=i} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} = \sum_i \binom{n-i}{k} \binom{k}{i}_{s-1} \frac{n+k}{n-i}.$$

□

5.2. Recurrence relations

Consider the sum of elements located along the direction (α, r) , with $r + \alpha > 0$, $r \in \mathbb{N}$, $0 \leq p < r$ and $\alpha \in \mathbb{Z}$ in the generalized s -Lucas triangle.

Let $(L_{n,s}^{(\alpha,\beta,r)})_n$ be the sequence obtain by this sum

$$L_{n,s}^{(\alpha,\beta,r)} := \sum_k \begin{bmatrix} n-rk \\ \beta + \alpha k \end{bmatrix}_{[s]}.$$

Theorem 5.5 For $n \geq \alpha s + r$, $(L_{n+1,s}^{(\alpha,\beta,r)})_n$ satisfies the following linear recurrence relation

$$\sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} L_{n-i,s}^{(\alpha,\beta,r)} = \sum_{i=0}^{\alpha(s-1)} \binom{\alpha}{i}_{s-1} L_{n-\alpha-r-i,s}^{(\alpha,\beta,r)}. \quad (5.4)$$

Proof

$$\begin{aligned} & \sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} L_{n-i,s}^{(\alpha,\beta,r)} \\ &= \sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} \sum_k \begin{bmatrix} n-rk-i \\ \beta + \alpha k \end{bmatrix}_{[s]} \\ &= \sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} \sum_k \begin{bmatrix} n-rk-i \\ \beta + \alpha k \end{bmatrix}_{[s]} + \sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} \sum_k \begin{bmatrix} n-rk-i-1 \\ \beta + \alpha k - 1 \end{bmatrix}_{[s]} \\ &+ 2 \sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} \sum_k \begin{bmatrix} n-rk-i-2 \\ \beta + \alpha k - 1 \end{bmatrix}_{[s]} + \cdots + s \sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} \sum_k \begin{bmatrix} n-rk-i-s \\ \beta + \alpha k - 1 \end{bmatrix}_{[s]}. \end{aligned}$$

By Theorem 4.3 we obtain

$$\begin{aligned} & \sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} L_{n-i,s}^{(\alpha,\beta,r)} \\ &= \sum_{i=0}^{\alpha(s-1)} \binom{\alpha}{i}_{s-1} \sum_k \binom{n-rk-i-r-\alpha}{\beta+\alpha k}_{[s]} + \sum_{i=0}^{\alpha(s-1)} \binom{\alpha}{i}_{s-1} \sum_k \binom{n-rk-i-r-\alpha-1}{\beta+\alpha k-1}_{[s]} \\ &+ 2 \sum_{i=0}^{\alpha(s-1)} \binom{\alpha}{i}_{s-1} \sum_k \binom{n-rk-i-r-\alpha-2}{\beta+\alpha k-1}_{[s]} + \cdots + s \sum_{i=0}^{\alpha(s-1)} \binom{\alpha}{i}_{s-1} \sum_k \binom{n-rk-i-r-\alpha-s}{\beta+\alpha k-1}_{[s]}. \end{aligned}$$

Finally by the relation (5.1) we get the result. $\square$

Corollary 5.6 *The sum of elements located along the direction (r, α) in the Lucas and Tribonacci-Lucas triangles are given respectively by*

$$\sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} L_{n-i,1}^{(\alpha,\beta,r)} = L_{n-\alpha-r,1}^{(\alpha,\beta,r)}, \quad (5.5)$$

$$\sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} L_{n-i,2}^{(\alpha,\beta,r)} = \sum_{i=0}^{\alpha} \binom{\alpha}{i} L_{n-\alpha-r-i,2}^{(\alpha,\beta,r)}. \quad (5.6)$$

Example 5.7 *The sum of elements located along the direction $(3, 2)$ in the Tribonacci-Lucas triangle ($s = 2$, $r = 3$, $\alpha = 2$ and $\beta = 0$) satisfies the following recurrence relation*

$$L_{n,2}^{(2,0,3)} = 3L_{n-1,2}^{(2,0,3)} - 3L_{n-2,2}^{(2,0,3)} + L_{n-3,2}^{(2,0,3)} + L_{n-5,2}^{(2,0,3)} + 3L_{n-6,2}^{(2,0,3)} + 3L_{n-7,2}^{(2,0,3)} + L_{n-8,2}^{(2,0,3)},$$

with $L_{0,2}^{(2,0,3)} = 3$, $L_{1,2}^{(2,0,3)} = 1$, $L_{2,2}^{(2,0,3)} = 1$, $L_{3,2}^{(2,0,3)} = 1$, $L_{4,2}^{(2,0,3)} = 1$, $L_{5,2}^{(2,0,3)} = 3$, $L_{6,2}^{(2,0,3)} = 15$, $L_{7,2}^{(2,0,3)} = 49$.

The first terms of this sequence are $(3, 1, 1, 1, 1, 3, 15, 49, 115, 221, 377, 611, 1027, 1935, \dots)$

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