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Two effective approaches for solving fractional generalized Hirota-Satsuma coupled KdV system arising in interaction of long waves

Ali Kurt, Hadi Rezazadeh, Mehmet Senol, Ahmad Neirameh, Orkun Tasbozan, Mostafa Eslami, Mohammad Mirzazadeh

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Highlights

- Summary the sub-equation method and power series method were successfully utilized to establish a series of new exact traveling wave solutions for the fractional generalized Hirota-Satsuma coupled KdV system.
- As a result, a series of new traveling wave solutions for the fractional generalized Hirota-Satsuma coupled KdV system were formally extracted. The performance of these methods are reliable, effective and computerized mathematical tool to handle nonlinear evolution equations in the field of mathematical physics and applied sciences.

Two effective approaches for solving fractional generalized Hirota-Satsuma coupled KdV system arising in interaction of long waves

Ali Kurt

Department of Mathematics, Mustafa Kemal University, Hatay, Turkey

E-mail: alikurt@mku.edu.tr

Hadi Rezazadeh

2Faculty of Engineering Technology, Amol University of Special Modern Technologies, Amol, Iran

E-mail: H.rezazadeh@ausmt.ac.ir

Mehmet Senol

Department of Mathematics, Nevsehir Hacı Bektas Veli University, Nevsehir, Turkey

E-mail: msenol@nevsehir.edu.tr

Ahmad Neirameh

Department of Mathematics, Faculty of Sciences, Gonbad Kavous University, Gonbad, Iran

E-mail: a.neirameh@gonbad.ac.ir

Orkun Tasbozan

Department of Mathematics, Mustafa Kemal University, Hatay, Turkey

E-mail: otasbozan@mku.edu.tr

Mostafa Eslami

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

E-mail: Mostafa.Eslami@umz.ac.ir

Mohammad Mirzazadeh

Department of Engineering Sciences, Faculty of Technology and Engineering, East of Guilan, University of Guilan, Rudsar-Vajargah, Iran

E-mail: Mirzazadehs2@guilan.ac.ir

ABSTRACT

In this article, two different methods, namely sub-equation method and residual power series method, have been used to obtain new exact and approximate solutions of the generalized Hirota-Satsuma system of equations, which is a coupled KdV model. The fractional derivative is taken in the conformable sense. Each of the obtained exact solutions were checked by substituting them into the corresponding system with the help of Maple symbolic computation package. The results indicate that both methods are easy to implement, effective and reliable. They are therefore ready to apply for various partial fractional differential equations.

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Key Words and Phrases: *Hirota-Satsuma coupled KdV system, Sub-equation method, power series method, conformable fractional derivative.*

1 Introduction

Fractional analysis is described as the expansion of integer-valued derivative and integral definitions of classical analysis to real, rational or complex scales. Although its history is nearly as old as classical integer-order analysis, the fractional calculus was not found practical applications in psychical sciences for many years. But the development of the symbolic computation programs like Matlab, Mathematica and Maple have caused the studies to accelerate. Thus a considerable effort has been expended and fractional calculus has been found many applications in fluid mechanics, viscoelasticity, electromagnetic, control theory, image processing, chemistry and biology. Nowadays a new well behaved, applicable and understandable definition of fractional derivative and integral are expressed by Khalil et. al. [31] called conformable fractional derivative and integral.

Definition 1.1. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a function. The α^{th} order "conformable fractional derivative" of f is defined by,

$$T_{\alpha}(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0, \alpha \in (0, 1)$.

Definition 1.2. If f is α -differentiable in some $(0, a), a > 0$ and $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$. The "conformable fractional integral" of a function f starting from $a \geq 0$ is defined as:

$$I_{\alpha}^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$.

Some properties and applications of conformable derivative are indicated in the works [1, 2, 8, 29, 31].

Besides ordinary and partial fractional differential equations arising in the above mentioned fields also attracted excessive attention. As a result many researchers have introduced a variety of powerful and reliable analytical and approximate methods to obtain solutions for fractional differential equations. These methods include, direct algebraic method [46–48, 50, 51, 53], function transformation method [32, 33], auxiliary equation method [52], variational method [25, 26, 34, 54], Hirota method [27], finite element method [18], ansatz method [13, 49], Adomian decomposition method [58], modified trial equation method [45], Lie symmetry method [14, 28], sine-Gordon equation method [6], first integral method [19–21, 43], sub-equation method [4, 9, 24], Riccati sub-equation method [35], extended trial function method [41, 44], simple equation method [5, 12, 15], homotopy analysis method [22, 23, 37–39] and perturbation-iteration algorithm [55–57].

In this article, we consider a long wave model. Waves in water shallower than $1/20$ their original wavelength are known as long waves or shallow-water waves. Many types of flow can be characterized as shallow-water equations. These equations describe an interaction of two long waves with different dispersion relations and they are well-suited for wave propagation phenomena.

Shallow water equations provide an interesting and rich field of research. Various implementation fields profit from shallow water theory. For instance, breakwater construction, wave propagation in tsunami forecasting, river flooding, coastal engineering, physical oceanography and dam break problems.

It is known that shallow flow equations are successful in calculating the effects of the floods spreading over large terrains and these equations are widely used for flood analysis. The analytical solutions of these equations also provide an accurate estimate of the coastal wave height and the time-dependent changes of the velocity distributions of shallow-water waves on a sloping beach and coastal areas. The height and velocity of the current occurring in the channels of a rainwater drainage network are expressed by shallow water wave equations. Moreover, they are used for calculating vorticity and flows in oceans.

Consequently, such a set of equations successfully addresses a large number of hydrodynamic problems.

The main purpose of this article is to obtain new exact and approximate solutions for the fractional generalized Hirota-Satsuma coupled KdV system as a long wave model, using the sub-equation and residual power series methods [10, 11, 42]. Sub-equation method is one of the reliable methods to obtain new exact solutions of partial fractional differential equations. Besides residual power series method is constructed on the power series expansion. It can be applied directly to the model without any transformation, perturbation, discretization or linearization by choosing suitable initial conditions.

Based on the methods, the analytical and approximate solitary wave solutions and periodic travelling wave solutions are obtained. These solutions are stated by the hyperbolic functions and the Jacobian elliptic functions.

2 Governing equations

In this article we apply the sub-equation method and residual power series method for solving fractional generalized Hirota-Satsuma coupled KdV system

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{1}{4} \frac{\partial^3 u}{\partial x^3} + 3u \frac{\partial u}{\partial x} + 3 \frac{\partial(-v^2 + w)}{\partial x} \\ \frac{\partial^\alpha v}{\partial t^\alpha} &= -\frac{1}{2} \frac{\partial^3 v}{\partial x^3} - 3u \frac{\partial v}{\partial x} \\ \frac{\partial^\alpha w}{\partial t^\alpha} &= -\frac{1}{2} \frac{\partial^3 w}{\partial x^3} - 3u \frac{\partial w}{\partial x},\end{aligned}\tag{2.1}$$

where

$$\frac{d^\alpha f}{dt^\alpha}(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad f : (0, \infty) \rightarrow \mathbb{R},\tag{2.2}$$

and $u = u(x, t)$, $v = v(x, t)$, $w = w(x, t)$ for all $t > 0$.

In recent years, many researchers have used this fractional derivative definition with different methods to obtain the exact and approximate solutions of various nonlinear partial fractional differential equations.

Hirota and Satsuma [27] presented Eq. (2.1) to describe interaction of two long waves with different dispersion relations. For $w = 0$, (2.1) reduces to the well-known generalized Hirota-Satsuma coupled KdV system [27]. This equation comes up with the 4-reduction of the KP hierarchy.

3 Description of Implemented Methods

3.1 Sub-Equation Method

In this part, we express a brief explanation of sub-equation method [59]. Consider the conformable fractional partial differential equation as

$$P(u, D_t^\alpha u, D_x u, D_t^{2\alpha} u, D_x^2 u, \dots) = 0\tag{3.1}$$

where $D_t^\alpha u$ means conformable fractional derivatives of function $u(x, t)$ and $D_t^{2\alpha}$ shows two times conformable fractional derivative of the an unknown function $u(x, t)$. By using the fractional wave transformation [16]

$$u(x, t) = U(\xi), \quad \xi = kx - p \frac{t^\alpha}{\alpha} \quad (3.2)$$

and the chain rule [1] Eq. (3.1) transforms into nonlinear ordinary differential equation

$$G(U, U', U'', \dots) = 0 \quad (3.3)$$

where prime indicates integer order derivative with respect to new wave variable ξ and k, p are arbitrary constants to be examined later. We suppose that equation (3.3) has a solution in the form

$$U(\xi) = \sum_{i=0}^N a_i \varphi^i(\xi), \quad a_N \neq 0, \quad (3.4)$$

where a_i ($0 \leq i \leq N$) are constant coefficients to be determined, N is a positive integer which is going to be found by balancing procedure [40] in equation (3.3) and $\varphi(\xi)$ is the solution of Riccati equation

$$\varphi'(\xi) = \sigma + (\varphi(\xi))^2 \quad (3.5)$$

where σ is an arbitrary constant. Some special solutions of the Riccati equation (3.5) are given by the following table.

$$\varphi(\xi) = \begin{cases} -\sqrt{-\sigma} \tanh(\sqrt{-\sigma}\xi), & \sigma < 0 \\ -\sqrt{-\sigma} \coth(\sqrt{-\sigma}\xi), & \sigma < 0 \\ \sqrt{\sigma} \tan(\sqrt{\sigma}\xi), & \sigma < 0 \\ \sqrt{\sigma} \cot(\sqrt{\sigma}\xi), & \sigma < 0 \\ -\frac{1}{\xi + \varpi}, & \varpi \text{ is a cons.}, \sigma = 0 \end{cases} \quad (3.6)$$

Substituting equations (3.4) and (3.5) into equation (3.3) we obtain a polynomial with respect $\varphi(\xi)$. Setting all the coefficients of $\varphi^i(\xi)$ ($i = 0, 1, \dots, N$) to zero, one gets nonlinear algebraic equation system in k, a_i ($i = 0, 1, \dots, N$). After solving these nonlinear algebraic equation system we determine the constants k, p, a_i ($i = 0, 1, \dots, N$). Then substituting obtained constants from the nonlinear algebraic system and the solutions of equation (3.5) into equation (3.4) by the help of the formulas (3.6) we acquire the exact solutions for equation (3.1).

3.2 Residual Power Series Method

In this section we are going to introduce some important definitions and theorems about residual power series

Theorem 3.1. Suppose that f is an infinitely α -differentiable function at a neighborhood of a point t_0 for some $0 < \alpha \leq 1$, then f has the fractional power series expansion of the form:

$$f(t) = \sum_{k=0}^{\infty} \frac{(T_\alpha^{t_0} f)^{(k)}(t_0)(t - t_0)^{k\alpha}}{\alpha^k k!}, \quad t_0 < t < t_0 + R^{\frac{1}{\alpha}}, \quad R > 0. \quad (3.7)$$

Here $(T_\alpha^{t_0} f)^{(k)}(t_0)$ represents the application of the fractional derivative k -times [1].

Definition 3.1. A multiple fractional power series about $t_0 = 0$ is defined by $\sum_{n=0}^{\infty} f_n(x)t^{n\alpha}$ for $0 \leq m - 1 < \alpha < m$, where t is a variable and $f_n(x)$ are functions called the coefficients of the series. [3, 17].

Theorem 3.2. Assume that $u(x, t)$ has a multiple fractional power series representation at $t_0 = 0$ of the form [3]

$$u(x, t) = \sum_{n=0}^{\infty} f_n(x) t^{n\alpha}, \quad 0 \leq m-1 < \alpha < m, \quad x \in I, \quad 0 \leq t \leq R^{\frac{1}{\alpha}}. \quad (3.8)$$

If $u_t^{(n\alpha)}(x, t)$, $n = 0, 1, 2, \dots$ are continuous on $I \times (0, R^{\frac{1}{\alpha}})$, then $f_n(x) = \frac{u_t^{(n\alpha)}(x, 0)}{\alpha^n n!}$.

To clarify the basic concept of RPSM, let's take a nonlinear fractional differential equation of the form:

$$T_\alpha u(x, t) + N[x]u(x, t) + R[x]u(x, t) = c(x, t), \quad x \in \mathbb{R}, \quad n-1 < n\alpha \leq n, \quad t > 0 \quad (3.9)$$

expressed by initial condition

$$f_0(x) = u(x, 0) = f(x) \quad (3.10)$$

where $R[x]$ is a linear, $N[x]$ is a non-linear operator and $c(x, t)$ are continuous functions. The RPS method made up of stating the solution of the equation (3.9) subject to (3.10) as a fractional power series expansion around $t = 0$.

$$f_{(n-1)}(x) = T_t^{(n-1)\alpha} u(x, 0) = h(x) \quad (3.11)$$

The expansion form of the solution is given by

$$u(x, t) = f(x) + \sum_{n=0}^{\infty} f_n(x) \frac{t^{n\alpha}}{\alpha^n n!} \quad (3.12)$$

In the next step, the k . truncated series of $u(x, t)$, namely $u_k(x, t)$ written as:

$$u_k(x, t) = f(x) + \sum_{n=0}^k f_n(x) \frac{t^{n\alpha}}{\alpha^n n!} \quad (3.13)$$

If the 1. RPSM approximate solution $u_1(x, t)$ is

$$u_1(x, t) = f(x) + f_1(x) \frac{t^\alpha}{\alpha^n} \quad (3.14)$$

then $u_k(x, t)$ could be reformulated as

$$u_k(x, t) = f(x) + f_1(x) \frac{t^\alpha}{\alpha^n} + \sum_{n=2}^k f_n(x) \frac{t^{n\alpha}}{\alpha^n n!} \quad (3.15)$$

for $0 < \alpha \leq 1$, $0 \leq t < R^{\frac{1}{\alpha}}$, $x \in I$ and $k = 2, 3, 4, \dots$

First we express the residual function as

$$Res(x, t) = T_\alpha u(x, t) + N[x]u(x, t) + R[x]u(x, t) - c(x, t) \quad (3.16)$$

and the k . residual function as

$$Res_k(x, t) = T_\alpha u_k(x, t) + N[x]u_k(x, t) + R[x]u_k(x, t) - g(x, t), \quad k = 1, 2, 3, \dots \quad (3.17)$$

It is clear that $Res(x, t) = 0$ and $\lim_{k \rightarrow \infty} Res_k(x, t) = Res(x, t)$ for each $x \in I$ and $0 \leq t$. In fact this

lead to $\frac{\partial^{(n-1)\alpha}}{\partial t^{(n-1)\alpha}} Res_k(x, t)$ for $n = 1, 2, 3, \dots, k$ because in the conformable sense, the fractional derivative of a constant is zero [7,30,36]. Solving the equation $\frac{\partial^{(n-1)\alpha}}{\partial t^{(n-1)\alpha}} Res_k(x, 0) = 0$ gives us the desired $f_n(x)$ coefficients. Thus the $u_n(x, t)$ approximate solutions can be obtained respectively. We can express the following theorem for the convergence analysis of the method [56].

Theorem 3.3. If there exists a fixed constant $0 < K < 1$ such that $\|u_{n+1}(x, y, t)\| \leq K \|u_n(x, y, t)\|$ for all $n \in \mathbb{N}$ and $0 < t < R < 1$, then the sequence of approximate solution converges to an exact solution.

Proof For all $0 < t < R < 1$, we have

$$\begin{aligned} & \|u(x, y, t) - u_n(x, y, t)\| \\ &= \left\| \sum_{i=n+1}^{\infty} u_i(x, y, t) \right\| \leq \sum_{i=n+1}^{\infty} \|u_i(x, y, t)\| \leq \|f(x, y)\| \sum_{i=n+1}^{\infty} K^i = \frac{K^{n+1}}{1-K} \|f(x, y)\| \xrightarrow{n \rightarrow \infty} 0. \end{aligned} \quad (3.18)$$

■

4 Solution of fractional generalized Hirota-Satsuma coupled KdV system

4.1 Application of Sub-equation Method

Considering Hirota-Satsuma coupled KdV system (2.1) then applying chain rule [1] and wave transform (3.2) the Eq. (2.1) turns into nonlinear ordinary differential equation system.

$$\begin{aligned} pu'(\xi) + \frac{1}{4}k^3u'''(\xi) + 3ku(\xi)u'(\xi) + 3k(-v^2(\xi) + w(\xi))' &= 0, \\ pv'(\xi) - \frac{1}{2}k^3v'''(\xi) - 3ku(\xi)v'(\xi) &= 0, \\ pw'(\xi) - \frac{1}{2}k^3w'''(\xi) - 3ku(\xi)w'(\xi) &= 0. \end{aligned} \quad (4.1)$$

Now regarding the solutions are represented by the following series

$$\begin{aligned} u(\xi) &= \sum_{i=0}^N a_i \varphi^i(\xi), \quad a_N \neq 0, \\ v(\xi) &= \sum_{i=0}^M b_i \varphi^i(\xi), \quad a_M \neq 0, \\ w(\xi) &= \sum_{i=0}^L b_i \varphi^i(\xi), \quad a_L \neq 0 \end{aligned} \quad (4.2)$$

where $\varphi(\xi)$ is the exact solutions of Riccati differential equation (3.5) given in (3.6). Using balancing procedure [40], we obtain $M = 2$, $L = 2$, $N = 2$. When we substitute all the obtained data into (4.1), the following algebraic equation system arises.

$$\begin{aligned}
& -3a_0c_1k\sigma - c_1k^3\sigma^2 + c_1p\sigma = 0, \\
& -6a_0c_2k\sigma - 3a_1c_1k\sigma - 8c_2k^3\sigma^2 + 2c_2p\sigma = 0, \\
& -3a_0c_1k - 6a_1c_2k\sigma - 3a_2c_1k\sigma - 4c_1k^3\sigma + c_1p = 0, \\
& -6a_0c_2k - 3a_1c_1k - 6a_2c_2k\sigma - 20c_2k^3\sigma + 2c_2p = 0, \\
& -6a_1c_2k - 3a_2c_1k - 3c_1k^3 = 0, \\
& -6a_2c_2k - 12c_2k^3 = 0, \\
& -3a_0b_1k\sigma - b_1k^3\sigma^2 + b_1p\sigma = 0, \\
& -6a_0b_2k\sigma - 3a_1b_1k\sigma - 8b_2k^3\sigma^2 + 2b_2p\sigma = 0, \\
& -3a_0b_1k - 6a_1b_2k\sigma - 3a_2b_1k\sigma - 4b_1k^3\sigma + b_1p = 0, \\
& -6a_0b_2k - 3a_1b_1k - 6a_2b_2k\sigma - 20b_2k^3\sigma + 2b_2p = 0, -6a_1b_2k - 3a_2b_1k - 3b_1k^3 = 0, \\
& -6a_2b_2k - 12b_2k^3 = 0, \\
& 3a_0a_1k\sigma + \frac{1}{2}a_1k^3\sigma^2 + a_1p\sigma - 6b_0b_1k\sigma + 3c_1k\sigma = 0, \\
& 6a_0a_2k\sigma + 3a_1^2k\sigma + 4a_2k^3\sigma^2 + 2a_2p\sigma - 12b_0b_2k\sigma - 6b_1^2k\sigma + 6c_2k\sigma = 0, \\
& 3a_0a_1k + 9a_1a_2k\sigma + 2a_1k^3\sigma + a_1p - 6b_0b_1k - 18b_1b_2k\sigma + 3c_1k = 0, \\
& 6a_0a_2k + 3a_1^2k + 6a_2^2k\sigma + 10a_2k^3\sigma + 2a_2p - 12b_0b_2k - 6b_1^2k - 12b_2^2k\sigma + 6c_2k = 0, \\
& 9a_1a_2k + \frac{3a_1k^3}{2} - 18b_1b_2k = 0, \\
& 6a_2^2k + 6a_2k^3 - 12b_2^2k = 0.
\end{aligned} \tag{4.3}$$

The following solution sets were obtained by solving the above equation system.

$$a_0 = \frac{p - 4k^3\sigma}{3k}, a_1 = 0, b_1 = 0, c_1 = 0, a_2 = -2k^2, b_2 = \mp k^2, b_0 = \mp \frac{3c_2 + 4k^4\sigma - 4kp}{6k^2}$$

For $\sigma < 0$

$$\begin{aligned}
u_1(x, t) &= \frac{p - 4k^3\sigma}{3k} + 2k^2\sigma \tanh \left(\sqrt{-\sigma} \left(kx - p\frac{t^\alpha}{\alpha} \right) \right)^2, \\
v_1(x, t) &= \mp \frac{3c_2 - 4kp + 4k^4\sigma}{6k^2} \pm k^2\sigma \tanh \left(\sqrt{-\sigma} \left(kx - p\frac{t^\alpha}{\alpha} \right) \right)^2, \\
w_1(x, t) &= c_0 - c_2\sigma \tanh \left(\sqrt{-\sigma} \left(kx - p\frac{t^\alpha}{\alpha} \right) \right)^2, \\
u_2(x, t) &= \frac{p - 4k^3\sigma}{3k} + 2k^2\sigma \coth^2 \left(\sqrt{-\sigma} \left(kx - p\frac{t^\alpha}{\alpha} \right) \right), \\
v_2(x, t) &= \mp \frac{3c_2 - 4kp + 4k^4\sigma}{6k^2} \pm k^2\sigma \coth \left(\sqrt{-\sigma} \left(kx - p\frac{t^\alpha}{\alpha} \right) \right)^2, \\
w_2(x, t) &= c_0 - c_2\sigma \coth \left(\sqrt{-\sigma} \left(kx - p\frac{t^\alpha}{\alpha} \right) \right)^2,
\end{aligned}$$

For $\sigma > 0$

$$u_3(x, t) = \frac{p - 4k^3\sigma}{3k} - 2k^2\sigma \tan^2 \left(\sqrt{\sigma} \left(kx - \frac{pt^\alpha}{\alpha} \right) \right),$$

$$\begin{aligned}
 v_3(x, t) &= \mp \frac{3c_2 + 4k^4\sigma - 4kp}{6k^2} \mp k^2\sigma \tan^2 \left(\sqrt{\sigma} \left(kx - \frac{pt^\alpha}{\alpha} \right) \right), \\
 w_3(x, t) &= c_0 + c_2\sigma \tan^2 \left(\sqrt{\sigma} \left(kx - \frac{pt^\alpha}{\alpha} \right) \right), \\
 u_4(x, t) &= \frac{p - 4k^3\sigma}{3k} - 2k^2\sigma \cot^2 \left(\sqrt{\sigma} \left(kx - \frac{pt^\alpha}{\alpha} \right) \right), \\
 v_4(x, t) &= \mp \frac{3c_2 + 4k^4\sigma - 4kp}{6k^2} \mp k^2\sigma \cot^2 \left(\sqrt{\sigma} \left(kx - \frac{pt^\alpha}{\alpha} \right) \right), \\
 w_4(x, t) &= c_0 + c_2\sigma \cot^2 \left(\sqrt{\sigma} \left(kx - \frac{pt^\alpha}{\alpha} \right) \right),
 \end{aligned}$$

For $\sigma = 0$

$$\begin{aligned}
 u_5(x, t) &= \frac{p}{3k} - \frac{2k^2}{\left(kx - \frac{pt^\alpha}{\alpha} + \omega\right)^2}, \\
 v_5(x, t) &= \mp \frac{3c_2 - 4kp}{6k^2} \mp \frac{k^2}{\left(kx - \frac{pt^\alpha}{\alpha} + \omega\right)^2}, \\
 w_5(x, t) &= c_0 + \frac{c_2}{\left(kx - \frac{pt^\alpha}{\alpha} + \omega\right)^2}.
 \end{aligned}$$

4.2 Application of Residual Power Series Method

Consider the nonlinear time fractional generalized Hirota-Satsuma coupled KdV system (2.1) with the initial conditions obtained from the exact solutions

$$u(x, 0) = f(x) = \frac{p}{3k} - \frac{2k^2}{(kx + \omega)^2} \quad (4.4)$$

$$v(x, 0) = g(x) = \frac{3c_2 - 4kp}{6k^2} + \frac{k^2}{(kx + \omega)^2} \quad (4.5)$$

$$w(x, 0) = h(x) = c_0 + \frac{c_2}{(kx + \omega)^2} \quad (4.6)$$

For residual power series

$$u(x, t) = f(x) + \sum_{n=0}^{\infty} f_n(x) \frac{t^{n\alpha}}{\alpha^n n!} \quad (4.7)$$

and k . truncated series of $u(x, t)$

$$u_k(x, t) = f(x) + \sum_{n=0}^k f_n(x) \frac{t^{n\alpha}}{\alpha^n n!}, \quad k = 1, 2, 3, \dots \quad (4.8)$$

The k_1 -th residual functions of time fractional HS equations are:

$$Resu_k(x, t) = \frac{\partial^\alpha u_k}{\partial t^\alpha} - \frac{1}{4} \frac{\partial^3 u_k}{\partial x^3} - 3u_k \frac{\partial u_k}{\partial x} - 3 \frac{\partial(-v_k^2 + w_k)}{\partial x} \quad (4.9)$$

$$Resv_k(x, t) = \frac{\partial^\alpha v_k}{\partial t^\alpha} + \frac{1}{2} \frac{\partial^3 v_k}{\partial x^3} + 3u_k \frac{\partial v_k}{\partial x} \quad (4.10)$$

$$Resw_k(x, t) = \frac{\partial^\alpha w_k}{\partial t^\alpha} + \frac{1}{2} \frac{\partial^3 w_k}{\partial x^3} + 3u_k \frac{\partial w_k}{\partial x} \quad (4.11)$$

To determine the coefficients $f_1(x)$, $g_1(x)$ and $h(x)$, in $u_k(x, t)$, $v_k(x, t)$ and $w_k(x, t)$ we should subrogate the 1st truncated series $u_1(x, t) = f(x) + f_1(x) \frac{t^\alpha}{\alpha}$, $v_1(x, t) = g(x) + g_1(x) \frac{t^\alpha}{\alpha}$ and $w_1(x, t) = h(x) + h_1(x) \frac{t^\alpha}{\alpha}$ into the 1st truncated residual functions as

$$\begin{aligned} Resu_1(x, t) = & f_1(x) - 3 \left(f(x) + \frac{t^\alpha f_1(x)}{\alpha} \right) \left(f'(x) + \frac{t^\alpha f_1'(x)}{\alpha} \right) \\ & - 3 \left(h'(x) - 2 \left(g(x) + \frac{t^\alpha g_1(x)}{\alpha} \right) \left(g'(x) + \frac{t^\alpha g_1'(x)}{\alpha} \right) + \frac{t^\alpha h_1'(x)}{\alpha} \right) \\ & + \frac{1}{4} \left(-f^{(3)}(x) - \frac{t^\alpha f_1^{(3)}(x)}{\alpha} \right) \end{aligned} \quad (4.12)$$

$$Resv_1(x, t) = g_1(x) + 3 \left(f(x) + \frac{t^\alpha f_1(x)}{\alpha} \right) \left(g'(x) + \frac{t^\alpha g_1'(x)}{\alpha} \right) + \frac{1}{2} \left(g^{(3)}(x) - \frac{t^\alpha g_1^{(3)}(x)}{\alpha} \right) \quad (4.13)$$

$$Resw_1(x, t) = h_1(x) + 3 \left(f(x) + \frac{t^\alpha f_1(x)}{\alpha} \right) \left(h'(x) + \frac{t^\alpha h_1'(x)}{\alpha} \right) + \frac{1}{2} \left(h^{(3)}(x) - \frac{t^\alpha h_1^{(3)}(x)}{\alpha} \right) \quad (4.14)$$

Now for the substitution of $t = 0$ through equations $Resu_1(x, t)$, $Resv_1(x, t)$ and $Resw_1(x, t)$ to obtain

$$Resu_1(x, 0) = f_1(x) - 3f(x)f'(x) - 3(h'(x) - 2g(x)g'(x)) - \frac{1}{4}f^{(3)}(x) \quad (4.15)$$

$$Resv_1(x, 0) = g_1(x) + 3f(x)g'(x) + \frac{1}{2}g^{(3)}(x) \quad (4.16)$$

$$Resw_1(x, 0) = h_1(x) + 3f(x)h'(x) + \frac{1}{2}h^{(3)}(x) \quad (4.17)$$

Thus for $Res_1(x, 0) = 0$

$$f_1(x) = 3f(x)f'(x) - 6g(x)g'(x) + 3h'(x) + \frac{1}{4}f^{(3)}(x) \quad (4.18)$$

$$g_1(x) = -3f(x)g'(x) - \frac{1}{2}g^{(3)}(x) \quad (4.19)$$

$$h_1(x) = -3f(x)h'(x) - \frac{1}{2}h^{(3)}(x) \quad (4.20)$$

Therefore, we obtain the 1st RPSM approximate solutions of time-fractional HS equation as

$$u_1(x, t) = f(x) + \frac{t^\alpha (12f(x)f'(x) - 24g(x)g'(x) + 12h'(x) + f^{(3)}(x))}{4\alpha} \quad (4.21)$$

$$v_1(x, t) = g(x) + \frac{t^\alpha (-6f(x)g'(x) - g^{(3)}(x))}{2\alpha} \quad (4.22)$$

$$w_1(x, t) = \frac{t^\alpha (-6f(x)h'(x) - h^{(3)}(x))}{2\alpha} \quad (4.23)$$

Again, to determine the second unknown coefficients $f_2(x)$, $g_2(x)$ and $h_2(x)$, we subrogate the 2nd truncated series solutions $u_2(x, t) = f(x) + f_1(x) \frac{t^\alpha}{\alpha} + f_2(x) \frac{t^{2\alpha}}{2\alpha^2}$, $v_2(x, t) = g(x) + g_1(x) \frac{t^\alpha}{\alpha} + g_2(x) \frac{t^{2\alpha}}{2\alpha^2}$ and $w_2(x, t) = h(x) + h_1(x) \frac{t^\alpha}{\alpha} + h_2(x) \frac{t^{2\alpha}}{2\alpha^2}$ into the 2nd truncated residual functions

to obtain

$$\begin{aligned} Resu_2(x, t) = & f_1(x) + \frac{t^\alpha f_2(x)}{\alpha} - 3 \left(f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha} f_2(x)}{2\alpha^2} \right) \left(f'(x) + \frac{t^\alpha f'_1(x)}{\alpha} + \frac{t^{2\alpha} f'_2(x)}{2\alpha^2} \right) \\ & - 3 \left(h'(x) - 2 \left(g(x) + \frac{t^\alpha g_1(x)}{\alpha} + \frac{t^{2\alpha} g_2(x)}{2\alpha^2} \right) \left(g'(x) + \frac{t^\alpha g'_1(x)}{\alpha} + \frac{t^{2\alpha} g'_2(x)}{2\alpha^2} \right) \right) \\ & + 3 \frac{t^\alpha h'_1(x)}{\alpha} + \frac{t^{2\alpha} h'_2(x)}{2\alpha^2} + \frac{1}{4} \left(-f^{(3)}(x) - \frac{t^\alpha f_1^{(3)}(x)}{\alpha} - \frac{t^{2\alpha} f_2^{(3)}(x)}{2\alpha^2} \right) \end{aligned} \quad (4.24)$$

$$\begin{aligned} Resv_2(x, t) = & g_1(x) + \frac{t^\alpha g_2(x)}{\alpha} + 3 \left(f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha} f_2(x)}{2\alpha^2} \right) \left(g'(x) + \frac{t^\alpha g'_1(x)}{\alpha} + \frac{t^{2\alpha} g'_2(x)}{2\alpha^2} \right) \\ & + \frac{1}{2} \left(g^{(3)}(x) + \frac{t^\alpha g_1^{(3)}(x)}{\alpha} + \frac{t^{2\alpha} g_2^{(3)}(x)}{2\alpha^2} \right) \end{aligned} \quad (4.26)$$

$$\begin{aligned} Resw_2(x, t) = & h_1(x) + \frac{t^\alpha h_2(x)}{\alpha} + 3 \left(f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha} f_2(x)}{2\alpha^2} \right) \left(h'(x) + \frac{t^\alpha h'_1(x)}{\alpha} + \frac{t^{2\alpha} h'_2(x)}{2\alpha^2} \right) \\ & + \frac{1}{2} \left(h^{(3)}(x) + \frac{t^\alpha h_1^{(3)}(x)}{\alpha} + \frac{t^{2\alpha} h_2^{(3)}(x)}{2\alpha^2} \right) \end{aligned} \quad (4.28)$$

Now, applying T_α on both sides of $Resu_2(x, t)$, $Resv_2(x, t)$ and $Resw_2(x, t)$ and equating to 0 for $t = 0$ gives:

$$f_2(x) = 3f_1(x)f'(x) - 6g_1(x)g'(x) + 3f(x)f'_1(x) - 6g(x)g'_1(x) + 3h'_1(x) + \frac{1}{4}f_1^{(3)}(x) \quad (4.29)$$

$$g_2(x) = -3f_1(x)g'(x) - 3f(x)g'_1(x) - \frac{1}{2}g_1^{(3)}(x) \quad (4.30)$$

$$h_2(x) = -3f_1(x)h'(x) - 3f(x)h'_1(x) - \frac{1}{2}h_1^{(3)}(x) \quad (4.31)$$

Therefore the 2nd RPSM approximate solutions of time-fractional HS equation is obtained as:

$$u_2(x, t) = f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha} \left(12(f_1(x)f'(x) - 2g_1(x)g'(x) + f(x)f'_1(x) - 2g(x)g'_1(x) + h'_1(x)) + f_1^{(3)}(x) \right)}{8\alpha^2} \quad (4.32)$$

$$v_2(x, t) = g(x) + \frac{t^\alpha g_1(x)}{\alpha} + \frac{t^{2\alpha} \left(-6f_1(x)g'(x) - 6f(x)g'_1(x) - g_1^{(3)}(x) \right)}{4\alpha^2} \quad (4.33)$$

$$w_2(x, t) = h(x) + \frac{t^\alpha h_1(x)}{\alpha} + \frac{t^{2\alpha} \left(-6f_1(x)h'(x) - 6f(x)h'_1(x) - h_1^{(3)}(x) \right)}{4\alpha^2} \quad (4.34)$$

In the same manner, we apply the same procedure for $n = 3$ to obtain the following results.

$$f_3(x) = 3f_2(x)f'(x) - 6g_2(x)g'(x) + 6f_1(x)f'_1(x) + 3f(x)f'_2(x) - 12g_1(x)g'_1(x) - 6g(x)g'_2(x) + 3h'_2(x) + \frac{1}{4}f_2^{(3)}(x) \quad (4.35)$$

$$\begin{aligned} u_3(x, t) = & f(x) + \frac{t^\alpha f_1(x)}{\alpha} + \frac{t^{2\alpha} f_2(x)}{2\alpha^2} + \frac{t^{3\alpha} (12f_2(x)f'(x) - 24g_2(x)g'(x) + 24f_1(x)f'_1(x))}{24\alpha^3} \\ & + \frac{t^{3\alpha} (12f(x)f'_2(x) - 48g_1(x)g'_1(x) - 24g(x)g'_2(x) + 12h'_2(x) + f_2^{(3)}(x))}{24\alpha^3} \end{aligned} \quad (4.36)$$

$$+ \frac{t^{3\alpha} (12f(x)f'_2(x) - 48g_1(x)g'_1(x) - 24g(x)g'_2(x) + 12h'_2(x) + f_2^{(3)}(x))}{24\alpha^3} \quad (4.37)$$

$$g_3(x) = -3f_2(x)g'(x) - 6f_1(x)g'_1(x) - 3f(x)g'_2(x) - \frac{1}{2}g_2^{(3)}(x) \quad (4.38)$$

$$v_3(x, t) = g(x) + \frac{t^\alpha g_1(x)}{\alpha} + \frac{t^{2\alpha} g_2(x)}{2\alpha^2} + \frac{t^{3\alpha} \left(-6f_2(x)g'(x) - 12f_1(x)g'_1(x) - 6f(x)g'_2(x) - g_2^{(3)}(x) \right)}{12\alpha^3} \quad (4.39)$$

$$h_3(x) = -3f_2(x)h'(x) - 6f_1(x)h'_1(x) - 3f(x)h'_2(x) - \frac{1}{2}h_2^{(3)}(x) \quad (4.40)$$

$$w_3(x, t) = h(x) + \frac{t^\alpha h_1(x)}{\alpha} + \frac{t^{2\alpha} h_2(x)}{2\alpha^2} + \frac{t^{3\alpha} \left(-6f_2(x)h'(x) - 12f_1(x)h'_1(x) - 6f(x)h'_2(x) - h_2^{(3)}(x) \right)}{12\alpha^3} \quad (4.41)$$

In Table 1, 2 and 3, the third order approximate RPSM solutions of time-fractional Hirota-Satsuma equations are compared numerically with exact solutions

$$u(x, t) = \frac{p}{3k} - \frac{2k^2}{(kx - \frac{t^\alpha p}{\alpha} + \omega)^2} \quad (4.42)$$

$$v(x, t) = \frac{3c_2 - 4kp}{6k^2} + \frac{k^2}{(kx - \frac{t^\alpha p}{\alpha} + \omega)^2} \quad (4.43)$$

$$w(x, t) = c_0 + \frac{c_2}{(kx - \frac{t^\alpha p}{\alpha} + \omega)^2} \quad (4.44)$$

for $\alpha = 0.25, \alpha = 0.50$ and $\alpha = 0.75$ and absolute errors are presented. The results indicate that as the t values increase the absolute errors also increase. Besides, as the α values increase, the absolute errors decrease. Also the tables show competitive results of the RPSM with highly approximate iterations. Moreover, in Figure 1, the surface plots of the approximate solutions are illustrated for $\alpha = 0.25, \alpha = 0.50$ and $\alpha = 0.75$.

Table 1: Comparison of third-order ($u(x, t)$) RPSM approximate and exact solutions with absolute errors for $\omega = 1, p = 0.5, k = 1, c_0 = 1.5, c_2 = 1.5$ and $x = 10$.

t	$\alpha = 0.25$			$\alpha = 0.50$			$\alpha = 0.75$		
	<i>RPSM</i>	<i>Exact</i>	<i>Abs. Error</i>	<i>RPSM</i>	<i>Exact</i>	<i>Abs. Error</i>	<i>RPSM</i>	<i>Exact</i>	<i>Abs. Error</i>
0.1	0.146169	0.146158	1.02893E-5	0.149145	0.149145	5.84622E-8	0.149776	0.149776	1.12961E-9
0.2	0.145266	0.145245	2.11327E-5	0.148707	0.148707	2.37353E-7	0.149522	0.149522	9.11829E-9
0.3	0.144631	0.144598	3.22811E-5	0.148361	0.14836	5.40252E-7	0.149295	0.149295	3.10190E-8
0.4	0.144121	0.144077	4.36608E-5	0.148061	0.14806	9.69945E-7	0.149082	0.149082	7.40672E-8
0.5	0.143688	0.143633	5.52325E-5	0.147790	0.147789	1.52885E-6	0.148879	0.148879	1.45668E-7
0.6	0.143307	0.143240	6.69709E-5	0.147541	0.147539	2.21916E-6	0.148683	0.148682	2.53387E-7
0.7	0.142964	0.142886	7.88581E-5	0.147307	0.147304	3.04290E-6	0.148491	0.148491	4.04951E-7
0.8	0.142651	0.142560	9.08807E-5	0.147086	0.147082	4.00199E-6	0.148304	0.148303	6.08237E-7
0.9	0.142361	0.142258	1.03028E-4	0.146875	0.14687	5.09824E-6	0.148119	0.148118	8.71279E-7
1.0	0.142091	0.141975	1.15292E-4	0.146673	0.146667	6.33340E-6	0.147937	0.147936	1.20226E-6

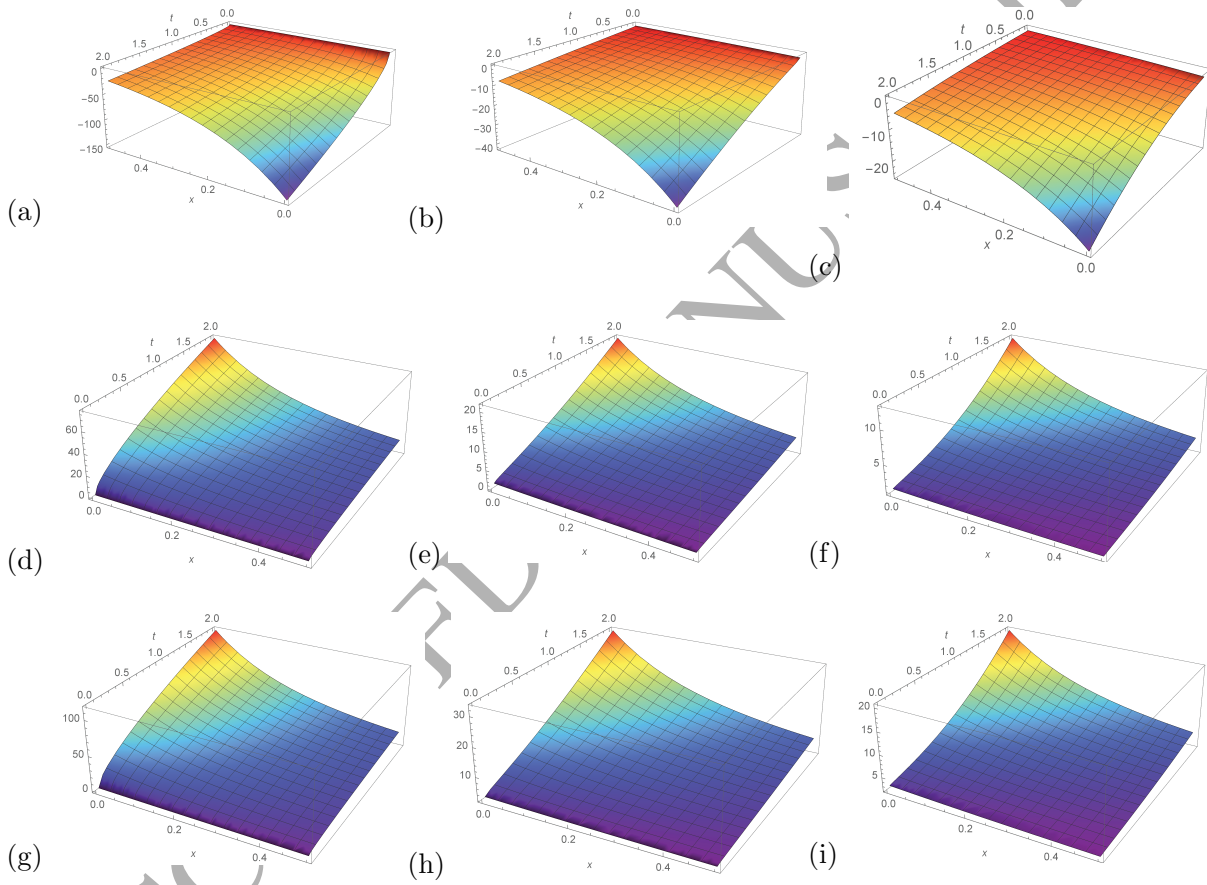


Figure 1: The surface plots for $\omega = 1$, $p = 0.5$, $k = 1$, $c_0 = 1.5$, $c_2 = 1.5$, and $x = 10$ of $u_3(x, t)$ for a.) $\alpha = 0.25$, b.) $\alpha = 0.50$, c.) $\alpha = 0.75$, of $v_3(x, t)$ for d.) $\alpha = 0.25$, e.) $\alpha = 0.50$, f.) $\alpha = 0.75$ and of $w_3(x, t)$ for g.) $\alpha = 0.25$, h.) $\alpha = 0.50$, i.) $\alpha = 0.75$.

Table 2: Comparison of third-order ($v(x, t)$) RPSM approximate and exact solutions with absolute errors for $\omega = 1$, $p = 0.5$, $k = 1$, $c_0 = 1.5$, $c_2 = 1.5$ and $x = 10$.

t	$\alpha = 0.25$			$\alpha = 0.50$			$\alpha = 0.75$		
	<i>RPSM</i>	<i>Exact</i>	<i>Abs. Error</i>	<i>RPSM</i>	<i>Exact</i>	<i>Abs. Error</i>	<i>RPSM</i>	<i>Exact</i>	<i>Abs. Error</i>
0.1	0.426916	0.426921	5.14466E-6	0.425428	0.425428	2.92311E-8	0.425112	0.425112	5.6480E-10
0.2	0.427367	0.427377	1.05664E-5	0.425646	0.425646	1.18677E-7	0.425239	0.425239	4.55914E-9
0.3	0.427685	0.427701	1.61405E-5	0.425820	0.425820	2.70126E-7	0.425353	0.425353	1.55095E-8
0.4	0.427939	0.427961	2.18304E-5	0.425970	0.425970	4.84973E-7	0.425459	0.425459	3.70336E-8
0.5	0.428156	0.428184	2.76162E-5	0.426105	0.426106	7.64426E-7	0.425561	0.425561	7.28339E-8
0.6	0.428346	0.428380	3.34854E-5	0.426230	0.426231	1.10958E-6	0.425659	0.425659	1.26694E-7
0.7	0.428518	0.428557	3.94290E-5	0.426346	0.426348	1.52145E-6	0.425754	0.425754	2.02475E-7
0.8	0.428674	0.428720	4.54403E-5	0.426457	0.426459	2.00100E-6	0.425848	0.425848	3.04119E-7
0.9	0.428819	0.428871	5.15141E-5	0.426562	0.426565	2.54912E-6	0.425940	0.425941	4.35640E-7
1.0	0.428955	0.429012	5.76460E-5	0.426663	0.426667	3.16670E-6	0.426031	0.426032	6.01131E-7

Table 3: Comparison of third-order ($w(x, t)$) RPSM approximate and exact solutions with absolute errors for $\omega = 1$, $p = 0.5$, $k = 1$, $c_0 = 1.5$, $c_2 = 1.5$ and $x = 10$.

t	$\alpha = 0.25$			$\alpha = 0.50$			$\alpha = 0.75$		
	<i>RPSM</i>	<i>Exact</i>	<i>Abs. Error</i>	<i>RPSM</i>	<i>Exact</i>	<i>Abs. Error</i>	<i>RPSM</i>	<i>Exact</i>	<i>Abs. Error</i>
0.1	1.51537	1.51538	7.71699E-6	1.51314	1.51314	4.38467E-8	1.51267	1.51267	8.4721E-10
0.2	1.51605	1.51607	1.58496E-5	1.51346	1.51347	1.78015E-7	1.51286	1.51286	6.83872E-9
0.3	1.51653	1.51655	2.42108E-5	1.51372	1.51373	4.05189E-7	1.51303	1.51303	2.32642E-8
0.4	1.51691	1.51694	3.27456E-5	1.51395	1.51396	7.27459E-7	1.51319	1.51319	5.55504E-8
0.5	1.51723	1.51728	4.14244E-5	1.51415	1.51416	1.14664E-6	1.51334	1.51334	1.09251E-7
0.6	1.51752	1.51757	5.02282E-5	1.51434	1.51435	1.66437E-6	1.51349	1.51349	1.90041E-7
0.7	1.51778	1.51784	5.91436E-5	1.51451	1.51452	2.28218E-6	1.51363	1.51363	3.03713E-7
0.8	1.51801	1.51808	6.81605E-5	1.51468	1.51469	3.00149E-6	1.51377	1.51377	4.56178E-7
0.9	1.51823	1.51831	7.72711E-5	1.51484	1.51485	3.82368E-6	1.51391	1.51391	6.53460E-7
1.0	1.51843	1.51852	8.64690E-5	1.51499	1.51500	4.75005E-6	1.51405	1.51405	9.01696E-7

5 Conclusion

In this article, sub-equation method and residual power series method was successfully employed to generalized coupled Hirota-Satsuma system. Firstly a series of new travelling wave solutions of the model are obtained. Traveling waves propagate in a certain direction with negligible change in shape. The existence of such waves began with the identification of the phenomena shown by hyperbolic equations. Fluid flow in the oceans and rivers, the progression of nerve impulses, chemical combustion, genetically modified (or cloned) organisms, and many other events in aerodynamics show a progressive wave behavior. Therefore, it is important to determine the dynamics of these equations. Then the approximate results are handled. The fractional derivatives are considered in the conformable sense. We have compared our exact and approximate results by tables and surface plots. We can conclude that both methods are simple, efficient and convenient. Thus they are ready to apply to wide range of physical problems arising in applied sciences.

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