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An efficient algorithm for solving fractional differential equations with boundary conditions

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Abstract: In this paper, a sinc-collocation method is described to determine the approximate solution of fractional order boundary value problem (FBVP). The results obtained are presented as two new theorems. The fractional derivatives are defined in the Caputo sense, which is often used in fractional calculus. In order to demonstrate the efficiency and capacity of the present method, it is applied to some FBVP with variable coefficients. Obtained results are compared to exact solutions as well as Cubic Spline solutions. The comparisons can be used to conclude that sinc-collocation method is powerful and promising method for determining the approximate solutions of FBVPs in different types of scenarios.

Keywords: Fractional differential equations, boundary value problems, sinc-collocation method, Caputo derivative

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1 Introduction

Many systems in applied sciences, such as, signal and image processing, earthquake engineering, electrochemistry and biomedical engineering, can be modeled by using fractional calculus in the form of fractional differential equations [29–32]. In order to better analyse these systems, it is requirement that the approximate solutions of these systems are known. For this, several solution methods have been developed to obtaining the approximate solutions of fractional differential equations. Some well-known methods for approximating solutions of FBVP are summarized as follows, but not limited to: Homotopy

perturbation method [1, 2], Differential transform method [3, 4], Adomian decomposition method [5–7], Variational iteration method [8, 9], Cubic spline method [10], Haar wavelet method [11] and Homotopy analysis method [12].

The sinc methods were introduced in [13], expanded in [14] by Frank Stenger and first analyzed in [15, 16], while sinc-collocation method for the numerical solution of the initial value problems was developed in [17]. Later, sinc-collocation method was studied by several authors in [18–23].

In this work, the sinc-collocation method is presented to obtain the approximate solution of a fractional order boundary value problem with variable coefficients in the following form

$$y'' + p(x)y' + q(x) {}^C D_x^\alpha y + r(x)y = f(x) \quad 0 < \alpha < 1, \quad (1.1)$$

and $y(a) = a_0, y(b) = b_0$

where ${}^C D_x^\alpha$ is the Caputo fractional derivative operator and m is an integer.

As an original contribution of the present paper to literature, in this study, the sinc-collocation method is firstly applied to solve the FBVPs. There is currently no study dealing with determination of the solutions of the FBVPs by using sinc-collocation method. The solution function is expanded to a finite series in terms of composite translated sinc functions and some unknown coefficients. After the integral resulting from the fractional term in the equation is approximately calculated using a sinc-quadrature rule where a conformal map and its inverse is evaluated at sinc grid points [24], these unknown coefficients are determined by present method. In order to show the ability and robustness of the sinc-collocation method, the method is applied to some specific FBVPs. Obtained results are compared with the exact ones and those obtained by the Cubic Spline method and hence the comparisons are shown. The results obtained comparisons indicate that the sinc-collocation method is a powerful and promising method for finding the approximate solutions of FBVPs.

The rest of this paper is organized as follows. In section 2, we have provided some definition and theorems for fractional calculus and the sinc-collocation method. In section 3, we have converted nonhomogeneous conditions

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given by (1.1) to homogeneous ones. In section 4, we use the sinc-collocation method to obtain an approximate solution of a general fractional differential equation and obtained results are given as two new theorems. In section 5, some test problems are given to show the ability of present method by using tables and graphics. Finally, in section 6, we have completed the paper with a conclusion.

2 Preliminaries

In this section, some definitions and theorems are given. For the sake of shortness, the reference of the proofs is given.

Definition 1. [25] Let $f : [a, b] \rightarrow \mathbb{R}$ be a function, α a positive real number, n the integer satisfying $n - 1 \leq \alpha < n$, and Γ the Euler gamma function. Then, the left Caputo fractional derivative of order α of $f(x)$ is given as

$${}_a^C D_x^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \int_a^x (x - t)^{n-\alpha-1} f^{(n)}(t) dt. \quad (2.1)$$

Definition 2. [19] The Sinc function is defined on the whole real line $-\infty < x < \infty$ by

$$\text{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0. \end{cases}$$

Definition 3. [19] For $h > 0$ and $k = 0, \pm 1, \pm 2, \dots$ the translated sinc function with space node are given by:

$$S(k, h)(x) = \text{sinc}\left(\frac{x - kh}{h}\right) = \begin{cases} \frac{\sin\left(\pi \frac{x - kh}{h}\right)}{\pi \frac{x - kh}{h}} & x \neq kh \\ 1 & x = kh. \end{cases}$$

Definition 4. [26] If $f(x)$ is defined on the real line, then for $h > 0$ the series

$$C(f, h)(x) = \sum_{k=-\infty}^{\infty} f(kh) \text{sinc}\left(\frac{x - kh}{h}\right)$$

is called the Whittaker cardinal expansion of f whenever this series converges.

In general, approximations can be constructed for infinite, semi-infinite and finite intervals. To construct approximation on the interval (a, b) the conformal map

$$\phi(z) = \ln\left(\frac{z - a}{b - z}\right) \quad (2.2)$$

is employed. This map carries D_E the eye-shaped domain in the z -plane

$$D_E = \left\{ z = x + iy : \left| \arg\left(\frac{z - a}{b - z}\right) \right| < d \leq \frac{\pi}{2} \right\}.$$

onto the infinite strip D_S

$$D_S \equiv \left\{ w = u + iv : |v| < d \leq \frac{\pi}{2} \right\}.$$

The basis functions on the interval (a, b) are derived from the composite translated sinc functions

$$S_k(z) = S(k, h)(z) \circ \phi(z) = \text{sinc}\left(\frac{\phi(z) - kh}{h}\right).$$

for $z \in D_E$. The inverse map of $w = \phi(z)$ is

$$z = \phi^{-1}(w) = \frac{a + be^w}{1 + e^w}.$$

The sinc grid points $z_k \in (a, b)$ in D_E will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \quad k = 0, \pm 1, \pm 2, \dots$$

Definition 5. [26] Let D_E be a simply connected domain in the complex plane $\mathbb{C}$, and let ∂D_E denote the boundary of D_E . Let a, b be points on ∂D_E and ϕ be a conformal map D_E onto D_S such that $\phi(a) = -\infty$ and $\phi(b) = \infty$. If the inverse map of ϕ is denoted by φ , define

$$\Gamma = \{\phi^{-1}(u) \in D_E : -\infty < u < \infty\}$$

and $z_k = \varphi(kh)$, $k = 0, \pm 1, \pm 2, \dots$

Definition 6. [27] Let $B(D_E)$ be the class of functions F that are analytic in D_E and satisfy

$$\int_{\psi(L+u)} |F(z)| dz \rightarrow 0, \text{ as } u = \mp \infty,$$

where

$$L = \left\{ iy : |y| < d \leq \frac{\pi}{2} \right\},$$

and those on the boundary of D_E satisfy

$$T(F) = \int_{\partial D_E} |F(z)| dz < \infty.$$

Theorem 1. Let Γ be $(0, 1)$, $F \in B(D_E)$, then for $h > 0$ sufficiently small,

$$\begin{aligned} \int_{\Gamma} F(z) dz - h \sum_{j=-\infty}^{\infty} \frac{F(z_j)}{\phi'(z_j)} &= \frac{i}{2} \int_{\partial D} \frac{F(z) k(\phi, h)(z)}{\sin(\pi \phi(z)/h)} dz \\ &\equiv I_F \end{aligned} \quad (2.3)$$

where

$$|k(\phi, h)|_{z \in \partial D} = \left| e^{\left[\frac{i \pi \phi(z)}{h} \text{sgn}(\text{Im} \phi(z)) \right]} \right|_{z \in \partial D} = e^{-\frac{\pi d}{h}}.$$

Proof: See [27]

For the term of fractional in (1.1), the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which an exponential convergence results.

Theorem 2. *If there exist positive constants α, β and C such that*

$$\left| \frac{F(x)}{\phi'(x)} \right| \leq C \begin{cases} e^{-\alpha|\phi(x)|} & x \in \psi((-\infty, \infty)) \\ e^{-\beta|\phi(x)|} & x \in \psi((0, \infty)). \end{cases} \quad (2.4)$$

then the error bound for the quadrature rule (2.3) is

$$\left| \int_{\Gamma} F(x) dx - h \sum_{j=-M}^N \frac{F(x_j)}{\phi'(x_j)} \right| \leq C \left(\frac{e^{-\alpha Mh}}{\alpha} + \frac{e^{-\beta Nh}}{\beta} \right) + |I_F|. \quad (2.5)$$

Proof: See [27].

The infinite sum in (2.3) is truncated with the use of (2.4) to arrive at the inequality (2.5). Making the selections

$$h = \sqrt{\frac{\pi d}{\alpha M}}, \quad \text{and}$$

$$N \equiv \left[\left\lfloor \frac{\alpha M}{\beta} + 1 \right\rfloor \right],$$

where $[\cdot]$ is an integer part of the statement and M is the integer value which specifies the grid size, then

$$\int_{\Gamma} F(x) dx = h \sum_{j=-M}^N \frac{F(x_j)}{\phi'(x_j)} + O\left(e^{-(\pi \alpha d M)^{1/2}}\right). \quad (2.6)$$

We used these theorems to approximate the arising integral in the formulation of the term fractional in (1.1).

Lemma 1. *Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_S , given by (2.2). Then*

$$\delta_{jk}^{(0)} = [S(j, h) \circ \phi(x)]|_{x=x_k} \begin{cases} 1 & j = k \\ 0 & j \neq k. \end{cases}$$

$$\delta_{jk}^{(1)} = h \frac{d}{d\phi} [S(j, h) \circ \phi(x)]|_{x=x_k} \begin{cases} 0 & j = k \\ \frac{(-1)^{k-j}}{k-j} & j \neq k. \end{cases}$$

$$\delta_{jk}^{(2)} = h^2 \frac{d^2}{d\phi^2} [S(j, h) \circ \phi(x)]|_{x=x_k} \begin{cases} -\frac{\pi^2}{3} & j = k \\ \frac{-2(-1)^{k-j}}{(k-j)^2} & j \neq k. \end{cases}$$

Proof: See [28].

3 Treatment of boundary conditions

Before mentioning the sinc-collocation method, we convert the nonhomogeneous conditions in (1.1) to homogeneous conditions by using the following transformation [19]:

$$u(x) = y(x) - H(x),$$

where

$$H(x) = \frac{b_0 - a_0}{b - a} x + \frac{b a_0 - a b_0}{b - a}.$$

Then problem (1.1) converts to the following boundary value problem with homogeneous boundary conditions

$$u''(x) + p(x)u'(x) + q(x) {}^C D_x^\alpha u(x) + r(x)u(x) = \hat{f}(x), \quad (3.1)$$

$$a < x < b \quad u(a) = u(b) = 0,$$

where

$$\hat{f} = f - (H'' + pH' + q {}^C D_x^\alpha H + rH).$$

4 The sinc-collocation method

We assume an approximate solution for $u(x)$ in problem (3.1) by the finite expansion of sinc basis functions

$$u_n(x) = \sum_{k=-M}^N c_k S_k(x), \quad n = M + N + 1 \quad (4.1)$$

where $S_k(x)$ is the function $S(k, h) \circ \phi(x)$. The unknown coefficients c_k in (4.1) are determined by sinc-collocation method. For this purpose, the first and second derivatives of $u_n(x)$ are given by

$$\frac{d}{dx} u_n(x) = \sum_{k=-M}^N c_k \phi'(x) \frac{d}{d\phi} S_k(x), \quad (4.2)$$

$$\frac{d^2}{dx^2} u_n(x) = \sum_{k=-M}^N c_k \left(\phi''(x) \frac{d}{d\phi} S_k(x) + (\phi')^2 \frac{d^2}{d\phi^2} S_k(x) \right) \quad (4.3)$$

Similarly, α order derivative of $u_n(x)$ for $0 < \alpha < 1$ is given by the following theorem.

Theorem 3. *If ξ is a conformal map for the interval $[a, x]$, then α order derivative of $u_n(x)$ for $0 < \alpha < 1$ is given by*

$${}_a^C D_x^\alpha (u_n(x)) \approx \sum_{k=-M}^N c_k R(x) \quad (4.4)$$

where

$$R(x) = \frac{h_L}{\Gamma(1-\alpha)} \sum_{r=-L}^L \frac{(x - x_r) S'_k(x_r)}{\xi'(x_r)}.$$

Proof: If we use the definition of Caputo fractional derivative given in (2.1), it is written that

$${}_a^C D_x^\alpha(u_n(x)) = \sum_{k=-M}^N c_k {}_a^C D_x^\alpha(S_k(x))$$

where

$${}_a^C D_x^\alpha(S_k(x)) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} S'_k(t) dt$$

Now we use the quadrature rule given in (2.6) to compute the above integral which is divergent on the interval $[a, x]$. For this purpose, a conformal map and its inverse image that denotes the sinc grid points are given by

$$\xi(t) = \ln\left(\frac{t-a}{x-t}\right)$$

and

$$x_r = \xi^{-1}(r h_L) = \frac{a + x e^{r h_L}}{1 + e^{r h_L}},$$

where $h_L = \pi/\sqrt{L}$. Then, according to equality (2.6), we write

$${}_a^C D_x^\alpha(S_k(x)) \approx \frac{h_L}{\Gamma(1-\alpha)} \sum_{r=-L}^L \frac{(x-x_r) S'_k(x_r)}{\xi'(x_r)}. \quad (4.5)$$

This completes the proof.

Replacing each term of (1.1) with the approximation given in (4.1)-(4.3), (4.5), multiplying the resulting equation by $\{(1/\phi')^2\}$ and setting $x = x_j$, we obtain the following linear system

$$\begin{aligned} & \sum_{k=-M}^N c_k \left\{ \frac{d^2}{d\phi^2} S_k + \left[p\left(\frac{1}{\phi'}\right) - \left(\frac{1}{\phi'}\right)' \right] \frac{d}{d\phi} S_k \right. \\ & \left. + q\left(\frac{1}{\phi'}\right)^2 R + r\left(\frac{1}{\phi'}\right)^2 S_k \right\} (x_j) \\ & = \left(\hat{f}\left(\frac{1}{\phi'}\right)^2 \right) (x_j), \quad j = -M, \dots, N. \end{aligned}$$

By using Lemma 1, we know that

$$\delta_{jk}^{(0)} = \delta_{kj}^{(0)}, \quad \delta_{jk}^{(1)} = -\delta_{kj}^{(1)}, \quad \delta_{jk}^{(2)} = \delta_{kj}^{(2)}$$

then following theorem becomes apparent.

Theorem 4. *If the assumed approximate solution of boundary value problem (1.1) is (4.1), then the discrete sinc-collocation system for the determination of the unknown co-*

efficients $\{c_k\}_{k=-M}^N$ is given by

$$\begin{aligned} & \sum_{k=-M}^N c_k \left\{ \frac{1}{h^2} \delta_{jk}^{(2)} + \frac{1}{h} \left[\left(\frac{1}{\phi'}\right)' - p\left(\frac{1}{\phi'}\right) \right] (x_j) \delta_{jk}^{(1)} \right. \\ & \left. + \left(q\left(\frac{1}{\phi'}\right)^2 R \right) (x_j) + \left(r\left(\frac{1}{\phi'}\right)^2 \right) (x_j) \delta_{jk}^{(0)} \right\} \\ & = \left(\hat{f}\left(\frac{1}{\phi'}\right)^2 \right) (x_j), \quad j = -M, \dots, N. \end{aligned} \quad (4.6)$$

Define some notation to represent in the matrix-vector form for system (3.7). Let $D(y)$ denotes a diagonal matrix whose diagonal elements are $y(x_{-M}), y(x_{-M+1}), \dots, y(x_N)$ and non-diagonal elements are zero, let $G = R(x_j)$ denote a matrix and also let $I^{(i)}$ denotes the matrices

$$I^{(i)} = [\delta_{jk}^{(i)}], \quad i = 0, 1, 2$$

where $D, G, I^{(0)}, I^{(1)}$ and $I^{(2)}$ are square matrices of order $n \times n$. Particularly, $I^{(0)}, I^{(1)}$ and $I^{(2)}$ are the identity matrix, the skew-symmetric matrix and the symmetric matrix, respectively. In order to calculate unknown coefficients c_k in linear system (4.6), we rewrite this system by using the above notations in matrix-vector form as

$$AC = B \quad (4.7)$$

where

$$\begin{aligned} A &= \frac{1}{h^2} I^{(2)} + \frac{1}{h} D \left(\left(\frac{1}{\phi'}\right)' - p\left(\frac{1}{\phi'}\right) \right) I^{(1)} + D \left(q\left(\frac{1}{\phi'}\right)^2 \right) G \\ &+ D \left(r\left(\frac{1}{\phi'}\right)^2 \right) I^{(0)} \\ B &= \left(\left(\hat{f}\left(\frac{1}{\phi'}\right)^2 \right) (x_{-M}), \left(\hat{f}\left(\frac{1}{\phi'}\right)^2 \right) (x_{-M+1}), \dots, \right. \\ &\quad \left. \left(\hat{f}\left(\frac{1}{\phi'}\right)^2 \right) (x_N) \right)^T \\ C &= (c_{-M}, c_{-M+1}, \dots, c_N)^T \end{aligned}$$

Now we have a linear system of n equations with n unknown coefficients given by (4.7). Using *Newton's method*, we can obtain the unknown coefficients c_k that are necessary for approximate solution in (4.1).

5 Computational examples

In this section, three problems that have homogeneous/nonhomogeneous boundary conditions will be

tested by using the present method via *Mathematica10* on a personal computer. In all the examples, we take $d = \pi/2$, $\alpha = \beta = 1/2$, $N = M$.

Example 1 Consider linear fractional boundary value problem in the following form [10]

$$y''(x) + {}^C_0 D_x^{0.5} y(x) + y(x) = f(x),$$

subject to the nonhomogeneous boundary conditions

$$y(0) = 1, \quad y(1) = 2,$$

where $f(x) = 3 + x^2 \left(\frac{x^{-0.5}}{\Gamma(2.5)} + 1 \right)$. The exact solution of this problem is $y(x) = x^2 + 1$. First we convert the nonhomogeneous boundary conditions to homogeneous conditions by using the transformation

$$u(x) = y(x) - x - 1$$

This change of variable yields the following boundary value problem:

$$u''(x) + {}^C_0 D_x^{0.5} u(x) + u^2(x) + 2(x+1)u(x) = g(x),$$

with homogeneous boundary conditions

$$u(0) = u(1) = 0,$$

where

$$g(x) = 2 + x^2 \left(\frac{x^{-0.5}}{\Gamma(2.5)} + 1 \right) - x - \frac{x^{0.5}}{\Gamma(0.5)}.$$

The numerical solutions which are obtained by using the sinc-collocation method (SCM) for this problem are presented in Table 1 and Table 2. In addition (Table 1), the errors are compared with the ones computed by using the Cubic Spline method (CSM) and the graphics of the exact and approximate solutions for different values of L and M are given in Figure 1 and Figure 2.

Example 2 Consider the linear fractional boundary value problem

$$y''(x) - x {}^C_0 D_x^{0.3} y(x) = f(x),$$

subject to the homogeneous boundary conditions

$$y(0) = 0, \quad y(1) = 0,$$

where $f(x) = -6x + 2 + \frac{6}{\Gamma(3.7)} x^{3.7} - \frac{2}{\Gamma(2.7)} x^{2.7}$. The exact solution of this problem is $y(x) = x^2(1-x)$. The numerical solutions which are obtained by using the present method for this problem are presented in Table 3 and Table 4. Additionally, the graphics of the exact and approximate solutions for different values of L and M are given in Figure 3 and Figure 4.

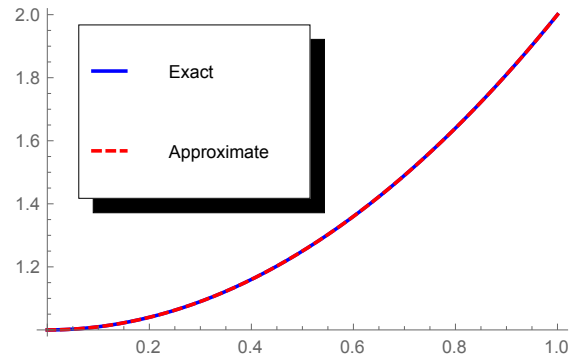


Figure 1: The graphics of the exact and approximate solutions for Example 1 when $L = 5$, $M = 5$

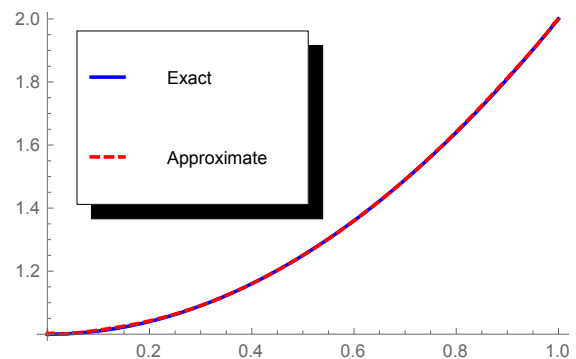


Figure 2: The graphics of the exact and approximate solutions for Example 1 when $L = 30$, $M = 50$

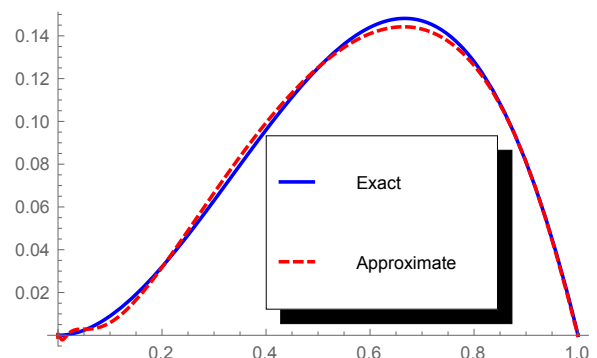


Figure 3: The graphics of the exact and approximate solutions for Example 2 when $L = 5$, $M = 5$

Example 3 Consider the following linear fractional boundary value problem:

$$y''(x) + x^2 y' - {}^C_0 D_x^{0.7} y(x) + y(x) = f(x)$$

Table 1: Numerical results for *Example 1* when $L = 5, M = 5$

x	Exact sol.	Approx Sol.(SCM)	Error(SCM)	Error(CSM)
0	1	1	0	0
0.125	1.01563	1.01886	3.23×10^{-3}	4.19×10^{-3}
0.250	1.06250	1.06371	1.20×10^{-3}	2.52×10^{-3}
0.375	1.14063	1.14069	6.21×10^{-5}	4.90×10^{-5}
0.500	1.25000	1.24991	9.10×10^{-5}	3.59×10^{-3}
0.625	1.39063	1.39056	6.63×10^{-5}	8.16×10^{-3}
0.750	1.56250	1.56348	9.82×10^{-4}	1.37×10^{-2}
0.875	1.76563	1.76864	3.01×10^{-3}	1.68×10^{-2}
1	2	2	0	0

Table 2: Numerical results for *Example 1* when $L = 30, M = 50$

x	Exact sol.	Approx Sol.(SCM)	Error(SCM)
0	1	1	0
0.125	1.01563	1.0156250	2.57×10^{-7}
0.250	1.06250	1.0625001	1.57×10^{-7}
0.375	1.14063	1.1406200	2.80×10^{-7}
0.500	1.25000	1.2499000	9.12×10^{-7}
0.625	1.39063	1.3906200	1.50×10^{-6}
0.750	1.56250	1.5624900	1.77×10^{-6}
0.875	1.76563	1.7656200	1.38×10^{-6}
1	2	2	0

Table 3: Numerical results for *Example 2* when $L = 5, M = 5$

x	Exact sol.	Approx Sol.	Error
0	0	0	0
0.1	0.009	0.00593671	3.06×10^{-3}
0.2	0.032	0.03169130	3.08×10^{-4}
0.3	0.063	0.06649110	3.49×10^{-3}
0.4	0.096	0.09924400	3.24×10^{-3}
0.5	0.125	0.12515000	1.49×10^{-4}
0.6	0.144	0.14100100	2.99×10^{-3}
0.7	0.147	0.14321300	3.78×10^{-3}
0.8	0.128	0.12633400	1.66×10^{-3}
0.9	0.081	0.08126190	2.61×10^{-4}
1	0	0	0

subject to the homogeneous boundary conditions

$$y(0) = 0, \quad y(1) = 0$$

where $f(x) = 5x^6 - 3x^5 - x^4 + 20x^3 - 12x^2 - \frac{120}{\Gamma(5.3)}x^{4.3} + \frac{24}{\Gamma(4.3)}x^{3.3}$. The exact solution of this problem is $y(x) = x^4(x - 1)$. The numerical solutions which are obtained by using the present method for this problem are presented in

Table 4: Numerical results for *Example 2* when $L = 30$, $M = 50$

x	Exact sol.	Approx Sol.	Error
0	0	0	0
0.1	0.009	0.0090000020	2.63×10^{-9}
0.2	0.032	0.0320000008	8.97×10^{-10}
0.3	0.063	0.0630000020	2.06×10^{-9}
0.4	0.096	0.0960000040	4.42×10^{-9}
0.5	0.125	0.1249999970	2.54×10^{-9}
0.6	0.144	0.1439999800	1.38×10^{-8}
0.7	0.147	0.1469999700	2.12×10^{-8}
0.8	0.128	0.1279999600	3.06×10^{-8}
0.9	0.081	0.0809999600	3.12×10^{-8}
1	0	0	0

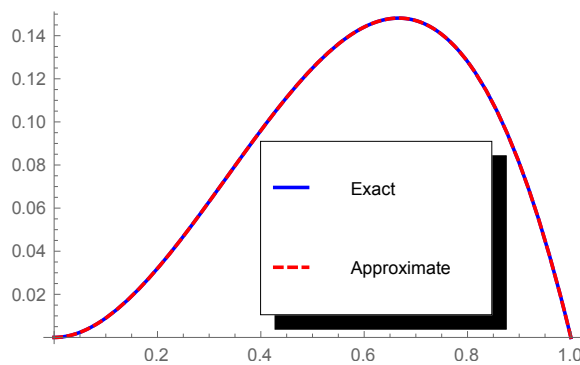
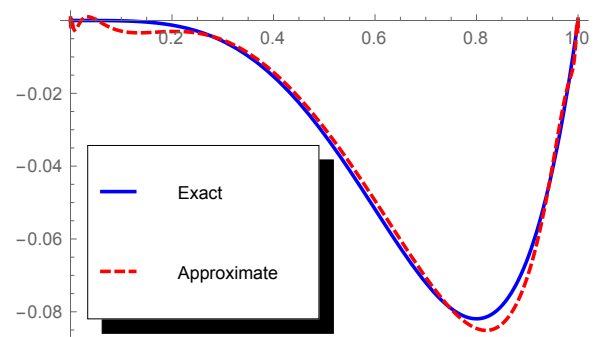
**Figure 4:** The graphics of the exact and approximate solutions for *Example 2* when $L = 30$, $M = 50$ **Figure 5:** The graphics of the exact and approximate solutions for *Example 3* when $L = 5$, $M = 5$

Table 5 and Table 6. Additionally, the graphics of the exact and approximate solutions for different values of L and M are given in Figure 5 and Figure 6.

6 Conclusion

In the present study, the sinc-collocation method is applied to find the approximate solutions of fractional order two-point boundary value problems. In order to illustrate the applicability and accuracy of the method for FBVPs, the method is applied to some special examples. Obtained solutions are compared with exact solutions and Cubic Spline solutions and differences are shown in tables and graphical forms. Observing these tabular and graphical forms, it can be concluded that sinc-collocation method is

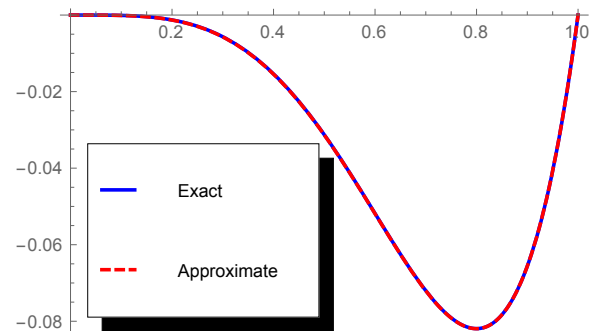
**Figure 6:** The graphics of the exact and approximate solutions for *Example 3* when $L = 30$, $M = 50$

Table 5: Numerical results for *Example 3* when $L = 5, M = 5$

x	Exact sol.	Approx Sol.	Error
0	0	0	0
0.1	-0.00009	-0.0029191	2.82×10^{-3}
0.2	-0.00128	-0.0030179	1.73×10^{-3}
0.3	-0.00567	-0.0053383	3.31×10^{-4}
0.4	-0.01536	-0.0142019	1.15×10^{-3}
0.5	-0.03125	-0.0294963	1.75×10^{-3}
0.6	-0.05184	-0.0494772	2.36×10^{-3}
0.7	-0.07203	-0.0705373	1.49×10^{-3}
0.8	-0.08192	-0.0845831	2.66×10^{-3}
0.9	-0.06561	-0.0704965	4.88×10^{-3}
1	0	0	0

Table 6: Numerical results for *Example 3* when $L = 30, M = 50$

x	Exact sol.	Approx Sol.	Error
0	0	0	0
0.1	-0.00009	-0.0001035	1.35×10^{-5}
0.2	-0.00128	-0.0013076	2.76×10^{-5}
0.3	-0.00567	-0.0057112	4.12×10^{-5}
0.4	-0.01536	-0.0154126	5.25×10^{-5}
0.5	-0.03125	-0.0313085	5.84×10^{-5}
0.6	-0.05184	-0.0518956	5.56×10^{-5}
0.7	-0.07203	-0.0720720	4.20×10^{-5}
0.8	-0.08192	-0.0819400	1.99×10^{-5}
0.9	-0.06561	-0.0656090	9.90×10^{-7}
1	0	0	0

very effective and powerful method for obtaining the approximate solution of FBVPs.

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