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New Analytical Solutions for Time Fractional Benjamin–Ono Equation Arising Internal Waves in Deep Water

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Abstract

In this article, the author sets up the abundant traveling wave solutions for time fractional Benjamin–Ono equation which was introduced to describe internal waves in stratified fluids by using Jacobi elliptic function expansion method. The traveling wave solutions are expressed in terms of the hyperbolic functions, the trigonometric functions and the rational functions. It can be seen that the obtained results are found to be important for the statement of some physical demonstrations of problems in mathematical physics and ocean engineering. In ocean engineering Benjamin–Ono equations are generally used in computer simulation for the water waves in deep water and open seas.

Key words: Jacobi elliptic function expansion method, Benjamin–Ono equation, conformable fractional derivative

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1 Introduction

The story of fractional calculus is as old and deeply routed as traditional calculus. Towards the end of the 17th century, it has come to exist with the letters between L'Hospital and Leibnitz. Then with the studies of the known mathematicians such as Laplace, Abel, Fourier, Liouville, Riemann, Grünwald and Letnikov, the basics of fractional calculus was developed. Actually fractional calculus can be considered as the generalized version of the known calculus. In fractional calculus the derivative and integral are described with arbitrary order. This description allows us to model the complex and nonlinear physical and engineering phenomena in the nature. So the obtained models can be well understood. To establish the mathematical models of the physical or engineering event scientists used different definitions of fractional derivative and integral such as Caputo derivative (Miller and Ross, 1993; Kilbas et al., 2006), Riemann−Liouville derivative (Podlubny, 1999), Kolwankar−Gangal (K−G) derivative (Kolwankar and Gangal, 1998), Chen's fractal derivative (Sun and Chen, 2009), Cresson's derivative (Cresson, 2003) and Jumarie's modified Riemann−Liouville derivative (Jumarie, 2009). Both of these definitions are used for various applications in different areas of science, for instance ocean modeling, plasma physics turbulence, stochastic dynamical system, image processing, fluid dynamics and astrophysics (Tasbozan et al., 2018; Çenesiz et al., 2017; Tasbozan et al., 2017; Kurt et al., 2017). Each derivative definition removes a deficiency of the other one or become the same of the other one under special conditions. For example Riemann−Lioville fractional derivative turns into Grunwald−Letnikov under some special conditions. Riemann−Liouville derivative uses boundary/initial conditions which are stated in Riemann−Liouville sense. But Caputo derivative uses boundary/initial conditions with integer order derivative. Also, The Riemann−Liouville derivative does not satisfy $D_a^\alpha = 0$ (Caputo derivative satisfies) where a is a constant if α is not a natural number. Riemann−Liouville and Caputo derivatives do not satisfy $D^\alpha D^\beta = D^{\alpha+\beta}$ in general. Moreover, neither Riemann−Lioville nor Caputo definitions satisfy chain rule, Leibniz formula, known formula for the derivative of the product of two functions, known formula for the derivative of the quotient of two functions and etc. as Khalil et al. (2014) claimed. Because of these deficiencies scientists decided to offer a new definition which overcomes the above mentioned situations. Khalil et al. (2014) stated a well behaved, applicable, understandable and efficient definition of fractional derivative and integral called conformable fractional derivative and integral.

Definition 1.1 Let $f : [0,\infty) \to \mathbb{R}$ be a function. Conformable fractional derivative of f is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0, 1)$. If f is α-differentiable in some $(0, a)$, $a > 0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0) =$

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$\lim_{t\to 0^+} f^{(\alpha)}(t)$.

The conformable fractional integral of a function f starting from $a \geqslant 0$ is expressed as:

$$I_\alpha^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}}\mathrm{d}x,$$

where the integral is the usual Riemann improper integral and $\alpha \in (0,\ 1]$.

The following theorem points out some properties which are satisfied by the conformable fractional derivative (Khalil et al., 2014).

Theorem 1.2 Let $\alpha \in (0,\ 1]$ and suppose f, g are α-differentiable at point $t > 0$.

Then

1. $T_\alpha(cf+dg) = cT_\alpha(f)+cT_\alpha(g)$ for all $a,b\in\mathbb{R}$.
2. $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p\in\mathbb{R}$.
3. $T_\alpha(\lambda) = 0$ for all constant functions $f(t) = \lambda$.
4. $T_\alpha(fg) = fT_\alpha(g)+gT_\alpha(f)$.
5. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.
6. If, in addition to f differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{\mathrm{d}f}{\mathrm{d}t}$.

This newly defined calculus has attracted a large number of scientists' interest. They used this derivative to make mathematical models of many physical, biological, engineering events. For example, Eslami and Rezazadeh (2018) used the multi-step conformable fractional differential transform method (MSCDTM) to give approximate solutions of the conformable fractional-order differential systems. Kumar et al. (2018) applied modified Kudryashov method to get the exact solutions of the fractional generalized reaction duffing (RD) model equation, the fractional biological population model and the fractional diffusion reaction (DR) equation. Kumar and Kaplan (2018) employed two efficient analytical techniques for obtaining the analytical solutions of conformable time fractional Zoomeron equation. Hosseini et al. (2018) studied the conformable time-fractional Klein−Gordon equations with quadratic and cubic nonlinearities and extracted several exact soliton solutions, including the bright (non-topological) and singular soliton solutions by making use of the ansatz method. Martinez et al. (2018) used conformable fractional derivative and obtained a new class of linear ordinary differential equations with non-integer power variable coefficients for the Resistance Capacitance (RC), Inductance Capacitance (LC), and Resistance, Inductance Capacitance (RLC) electric circuits. Mozaffari et al. (2018) introduced a conformable fractional quantum mechanics which has been introduced by using three postulates. Al-Shawba et al. (2018) used the G'/G-expansion method to construct the closed form solutions of the seventh order time fractional Sawada−Kotera−Ito (TFSKI) equation based on conformable fractional derivative. Yavuz and Ozdemir (2018) obtained semi analytical solutions of fractional Cauchy problem by using q-Homotopy Analysis Method (q-HAM) featuring conformable derivative.

To obtain better review insight for this subject one can easily see the references expressed below. Kurt (2019) obtained the exact solutions of fractional modified Camassa–Holm equation which is completely integrable dispersive shallow-water equation with the aid of Jacobi elliptic expansion method. Yaslan (2018) expressed the new analytic solutions of the space-time fractional Broer–Kaup and approximate long water wave equations. Mousa (2018) simulated a nonlinear shallow-water model of tsunami wave propagation numerically simulated using the method of lines. Kurt et al. (2019) employed sub equation method and residual power series method to obtain the wave solutions of generalized Hirota-Satsuma coupled KdV system arising in interaction of long waves. Gao (2015) studied a generalized (3+1)-dimensional variable-coefficient BKP equation for the nonlinear waves in fluid mechanics and acquired the exact results for this equation.

The rest of article is organized as follows. In Section 2 we give some information about the considered model, in Section 3 description of the method is expressed. In Section 4 application of the method is stated. In Section 5 graphical illustrations of some wave solutions are given.

2 Model under investigation

Benjamin–Ono (BO) equation (Benjamin, 1967) which looks likes KDV equation was expressed to explain internal waves in stratified fluids. It has also been used to model surface wave propagation on a thin layered structure (Ewen et al., 1980), using a surface acoustic wave delay line to launch the waves. In this study, we consider the conformable time fractional BO equation

$$\mathrm{D}_t^{(2\alpha)}u+\beta\mathrm{D}_x^2(u^2)+\gamma\mathrm{D}_x^4u = 0, \tag{1}$$

where γ and β are nonzero constants and $\mathrm{D}_t^{(2\alpha)}$ means two times conformable fractional derivative of function $u(x,t)$ with respect to variable t. By means of Jacobi elliptic function expansion method, the exact solutions of the BO equation which is one of the important nonlinear models in physics were obtained.

3 Brief description of the Jacobi elliptic function expansion method

Now let us identify the basics of Jacobi elliptic function expansion method step by step.

Step 1:

Conceive the following nonlinear conformable time fractional equation,

$$F\left(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial u}{\partial x}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^2 u}{\partial x^2}, \ldots\right) = 0. \tag{2}$$

Using the wave transformation where w is the arbitrary constants to be determined later,

$$u=u(\xi),\xi=x+w\frac{t^{\alpha}}{\alpha} \tag{3}$$

to Eq. (2) then employing the chain rule (Abdeljawad, 2015)

$$\frac{\partial^{\alpha}(.)}{\partial t^{\alpha}}=m\frac{\mathrm{d}(.)}{\mathrm{d}\xi},\ \frac{\partial(.)}{\partial x}=n\frac{\mathrm{d}(.)}{\mathrm{d}\xi},\ \frac{\partial(.)}{\partial y}=s\frac{\mathrm{d}(.)}{\mathrm{d}\xi},\dots \tag{4}$$

yields an ordinary differential equation (ODE)

$$O(u,u',u'',u''',\dots), \tag{5}$$

where prime indicates the integer order derivative with respect to variable ξ.

Step 2:

In Jacobi elliptic function expansion method $u(\xi)$ can be considered as a finite series

$$u(\xi)=\sum_{i=0}^{k}a_iF^i(\xi), \tag{6}$$

where $F(\xi)$ is the solution of the following nonlinear ordinary equation

$$(F')^2(\xi)=PF^4(\xi)+QF^2(\xi)+R. \tag{7}$$

In this differential equation, $F'=\mathrm{d}F/\mathrm{d}\xi$, $\xi=\xi(x,t,y)$, and P, Q, and R are constants chosen from Table 1. Eq. (7) has the following solutions shown in Table 1 where $\mathrm{i}^2=-1$. The denoted functions in Table 1 are called the Jacobian elliptic functions $\mathrm{sn}\xi=\mathrm{sn}(\xi;r)$, $\mathrm{cn}\xi=\mathrm{cn}(\xi;r)$ and $\mathrm{dn}\xi=\mathrm{dn}(\xi;r)$ where $r(0<r<1)$ is the modulus of the elliptic function, which are double periodic and satisfy the following properties.

$$\mathrm{sn}^2\xi+\mathrm{cn}^2\xi=1;$$
$$\mathrm{dn}^2\xi+r^2\mathrm{sn}^2\xi=1;$$
$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{sn}\xi=\mathrm{cn}\xi\mathrm{dn}\xi;$$

Table 1 Solutions of Eq. (7) for the chosen values of Q and R (Lai et al., 2009; Li, 2005)

	P	Q	R	F
1	r^2	$-(1+r^2)$	1	sn,cd
2	$-r^2$	$2r^2-1$	$1-r^2$	cn
3	-1	$2-r^2$	r^2-1	dn
4	1	$-(1+r^2)$	r^2	ns,dc
5	$1-r^2$	$2r^2-1$	$-r^2$	nc
6	r^2-1	$2-r^2$	-1	nd
7		$2-r^2$	1	sc
8	$r^2(1-r^2)$	$2r^2-1$	1	sd
9	1	$2-r^2$	$1-r^2$	cs
10	1	$2r^2-1$	$-r^2(1-r^2)$	ds
11	$-\frac{1}{4}$	$\frac{r^2+1}{2}$	$-\frac{(1-r^2)^2}{4}$	$r(\mathrm{cn}\mp\mathrm{dn})$
12	$\frac{1}{4}$	$\frac{-2r^2+1}{2}$	$\frac{1}{4}$	$\mathrm{ns}\mp\mathrm{cs}$
13	$\frac{1-r^2}{4}$	$\frac{r^2+1}{2}$	$\frac{1-r^2}{4}$	$\mathrm{ns}\mp\mathrm{sc}$
14	$\frac{1}{4}$	$\frac{r^2-2}{2}$	$\frac{r^4}{4}$	$\mathrm{ns}\mp\mathrm{ds}$
15	$\frac{r^4}{4}$	$\frac{r^2-2}{2}$	$\frac{r^4}{4}$	$\mathrm{sn}\mp\mathrm{i}\,\mathrm{cn},\ \frac{\mathrm{dn}}{\sqrt{1-r^2}\mathrm{sn}\mp\mathrm{cn}}$
16	$\frac{1}{4}$	$\frac{1-2r^2}{2}$	$\frac{1}{4}$	$r\,\mathrm{cn}\mp\mathrm{i}\,\mathrm{dn},\ \frac{\mathrm{sn}}{1\mp\mathrm{cn}}$
17	$\frac{r^2}{4}$	$\frac{r^2-2}{2}$	$\frac{1}{4}$	$\frac{\mathrm{sn}}{1\mp\mathrm{dn}}$
18	$\frac{r^2-1}{4}$	$\frac{r^2+1}{2}$	$\frac{r^2-1}{4}$	$\frac{\mathrm{dn}}{1\mp r\,\mathrm{sn}}$
19	$\frac{1-r^2}{4}$	$\frac{r^2+1}{2}$	$\frac{-r^2+1}{4}$	$\frac{\mathrm{cn}}{1\mp\mathrm{sn}}$
20	$\frac{(1-r^2)^2}{4}$	$\frac{r^2+1}{2}$	$\frac{1}{4}$	$\frac{\mathrm{sn}}{\mathrm{dn}\mp\mathrm{cn}}$
21	$\frac{r^4}{4}$	$\frac{r^2-2}{2}$	$\frac{1}{4}$	$\frac{\mathrm{cn}}{\sqrt{1-r^2}\mp\mathrm{dn}}$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{cn}\xi = -\mathrm{sn}\xi\mathrm{dn}\xi;$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{dn}\xi = -r^2\mathrm{cn}\xi\mathrm{sn}\xi.$$

The main idea in this method is to use the occasion of the solutions of the auxiliary ordinary differential Eq. (7) to construct abundant families of Jacobian elliptic solutions.

Since $r \to 0$ and $r \to 1$, Jacobi elliptic functions transform to trigonometric and hyperbolic functions as shown in Table 2, and thus we acquire the trigonometric function solutions and solitonic solutions of the considered equation.

Table 2 Jacobi elliptic functions for $r \to 0$ and $r \to 1$ (Lai et al., 2009; Li, 2005).

		$r \to 0$	$r \to 1$			$r \to 0$	$r \to 1$
1	sn$u$	sin$u$	tanh$u$	7	dc$u$	sec$u$	1
2	cn$u$	cos$u$	sech$u$	8	nc$u$	sec$u$	cosh$u$
3	dn$u$	1	sech$u$	9	sc$u$	tan$u$	sinh$u$
4	cd$u$	cos$u$	1	10	ns$u$	csc$u$	coth$u$
5	sd$u$	sin$u$	sinh$u$	11	ds$u$	csc$u$	csch$u$
6	nd$u$	1	cosh$u$	12	cs$u$	cot$u$	csch$u$

Step 3:

For the solution the constants $k, a_i (i = 0, 1, 2, \ldots, k)$ are going to be identified too. For this purpose, we can use balancing procedure. This procedure depends on balancing the highest order linear term

$$O\left(\frac{\mathrm{d}^p u}{\mathrm{d}\xi^p}\right) = k + p, \quad p= 0,1,2,... \tag{8}$$

and the highest order nonlinear term

$$O\left(u^q \frac{\mathrm{d}^p u}{\mathrm{d}\xi^p}\right) = (q+1)k + p, \quad p,q = 0,1,..., \tag{9}$$

in Eq. (5). So the integer k in Eq. (6) can be determined, substituting Eq. (6) and setting all the coefficients of powers F to be zero, and then a system of nonlinear algebraic equations for $a_i (i = 0, 1, 2, \ldots, n)$ is derived. After solving this system with the aid of a computer software and using all the values for P, Q, and R in Eq. (7) in Table 1 we can obtain the values for $a_i (i = 0, 1, 2, \ldots, n)$ and w.

Step 4:

Finally, we can combine the values with Eq. (6) and the auxiliary equation we choose, and obtain exact solutions for Eq. (2).

4 Formation of the solutions of time fractional Benjamin–Ono equation

Supposing the time fractional Benjamin–Ono equation (1) where $\alpha \in (0, 1)$ and γ, β are constants and fractional derivatives are in conformable sense. Employing the wave transformation Eq. (3) to Eq. (1) and then integrating twice leads to

$$w^2 u + \beta u^2 + \gamma u'' = 0, \tag{10}$$

where prime indicates the derivative of function $u(\xi)$ with regard to ξ. Using the balancing procedure between the highest order linear term and highest order nonlinear term we acquire $k = 2$. Hence the solution of Eq. (10) can be stated as:

$$u(\xi) = a_0 + a_1 F(\xi) + a_2 F^2(\xi). \tag{11}$$

Differentiating Eq. (11) twice leads to

$$u_{\xi\xi} = a_1 F''(\xi) + 2a_2 \left\{[F'(\xi)]^2 + F(\xi)F''(\xi)\right\} \tag{12}$$

then by Eq. (7)

$$F'' = 2PF^3 + QF. \tag{13}$$

Placing Eqs. (13) and (7) into Eq. (12) leads to

$$u_{\xi\xi} = a_1\left(2PF^3 + QF\right) + 2a_2\left(PF^4 + QF^2 + R\right) + 2a_2 F\left(2PF^3 + QF\right). \tag{14}$$

Subrogating Eq. (11) and Eq. (14) in Eq. (10) and regarding each coefficient of F to be zero, leads to the following equation system

$$a_0\left(a_0\beta + w^2\right) + 2a_2\gamma R = 0;$$
$$a_1\left(2a_0\beta + \gamma Q + w^2\right) = 0;$$
$$a_2\left(2a_0\beta + 4\gamma Q + w^2\right) + a_1^2\beta = 0;$$
$$2a_1(a_2\beta + \gamma P) = 0;$$
$$a_2(a_2\beta + 6\gamma P) = 0.$$

Solving the above system with the help of mathematics, we obtain the following solution set

$$a_0 = \frac{2\left[\beta\gamma(-Q) - \sqrt{\beta^2\gamma^2 Q^2 - 3\beta^2\gamma^2 PR}\right]}{\beta^2};$$

$$a_1 = 0;$$

$$a_2 = -\frac{6\gamma P}{\beta};$$

$$w = \pm\frac{2\sqrt[4]{\beta^2\gamma^2(Q^2 - 3PR)}}{\sqrt{\beta}}.$$

Finally combining the values with Eq. (11), we can obtain exact solutions of Eq. (1) by choosing the suitable values from Table 1 and Table 2 as follows.

When $P = r^2$, $Q = -(1 + r^2)$, $R = 1$ are chosen, $F = \mathrm{sn}$

from Table 1 and since $r \to 1$ from Table 2, solution can be written as:

$$u_1(x,t) = \frac{2\left(2\beta\gamma - \sqrt{\beta^2\gamma^2}\right)}{\beta^2} - \frac{6\gamma \tanh^2\left(x - \dfrac{2\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right)}{\beta}.$$

Setting $P = -r^2$, $Q = 2r^2 - 1$, $R = 1 - r^2$, from Table 1 $F = \mathrm{cn}$, due to this settings and when $r \to 1$, the solution can be obtained as:

$$u_2(x,t) = \frac{2\left(-\sqrt{\beta^2\gamma^2} - \beta\gamma\right)}{\beta^2} + \frac{6\gamma \,\mathrm{sech}^2\left(x - \dfrac{2\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right)}{\beta}.$$

Supposing $P = 1$, $Q = -(1 + r^2), R = r^2$, from Table 1 since $r \to 0$, the wave solution can be obtained

$$u_3(x,t) = \frac{2\left(\beta\gamma - \sqrt{\beta^2\gamma^2}\right)}{\beta^2} - \frac{6\gamma \csc^2\left(x - \dfrac{2\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right)}{\beta}.$$

For the same assumptions of P, Q, R and since $r \to 1$, the solution can be written

$$u_4(x,t) = \frac{2\left(2\beta\gamma - \sqrt{\beta^2\gamma^2}\right)}{\beta^2} - \frac{6\gamma \coth^2\left(x - \dfrac{2\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right)}{\beta}.$$

Again the same choices for $r \to 0$, solution can be declared as follows:

$$u_5(x,t) = \frac{2\left(\beta\gamma - \sqrt{\beta^2\gamma^2}\right)}{\beta^2} - \frac{6\gamma \sec^2\left(x - \dfrac{2\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right)}{\beta}.$$

While $P = 1 - r^2$, $Q = 2 - r^2$, $R = 2$, it can be deducted from Table 1 $F = \mathrm{sc}$, then the solution can be evaluated when $r \to 0$

$$u_6(x,t) = \frac{2\left(-\sqrt{\beta^2\gamma^2} - 2\beta\gamma\right)}{\beta^2} - \frac{6\gamma \tan^2\left(x - \dfrac{2\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right)}{\beta}.$$

Also assigning $P = 1, Q = 2 - r^2, R = 1 - r^2$ and $F = \mathrm{cs}$ from Table 1, the solution is found as if $r \to 0$

$$u_7(x,t) = \frac{2\left(-\sqrt{\beta^2\gamma^2} - 2\beta\gamma\right)}{\beta^2} - \frac{6\gamma \cot^2\left(x - \dfrac{2\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right)}{\beta}.$$

Furthermore, if $r \to 1$ for the same choices, the wave solution

$$u_8(x,t) = \frac{2\left(-\sqrt{\beta^2\gamma^2} - \beta\gamma\right)}{\beta^2} - \frac{6\gamma \,\mathrm{csch}^2\left(x - \dfrac{2\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right)}{\beta}$$

is obtained. As $P = \dfrac{1}{4}, Q = \dfrac{-2r^2 + 1}{2}$, $R = \dfrac{1}{4}$ and $F = \mathrm{ns} \pm \mathrm{cs}$ from Table 1, the solution can be evaluated while $r \to 0$

$$u_{9,10}(x,t) = \frac{2\left(-\dfrac{1}{4}\sqrt{\beta^2\gamma^2} - \dfrac{\beta\gamma}{2}\right)}{\beta^2} - \frac{3\gamma\left[\cot\left(x - \dfrac{\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right) \pm \csc\left(x - \dfrac{\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right)\right]^2}{2\beta}.$$

If we choose P, Q, R as $P = \dfrac{1}{4}, Q = \dfrac{-2r^2 + 1}{2}$, $R = \dfrac{1}{4}$, from Table 1 F is obtained as $F = \mathrm{ns} \mp \mathrm{cs}$, in this way the solution can be expressed as when $r \to 1$

$$u_{11,12}(x,t) = \frac{2\left(\dfrac{\beta\gamma}{2} - \dfrac{1}{4}\sqrt{\beta^2\gamma^2}\right)}{\beta^2} - \frac{3\gamma\left[\coth\left(x - \dfrac{\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right) \pm \mathrm{csch}\left(x - \dfrac{\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right)\right]^2}{2\beta}.$$

Set $P = \dfrac{1 - r^2}{4}$, $Q = \dfrac{r^2 + 1}{2}$, $R = \dfrac{1 - r^2}{4}$, from Table 1 $F = \mathrm{nc} \mp \mathrm{sc}$, due to this settings while $r \to 1$, the solution can be expressed as:

$$u_{13,14}(x,t) = \frac{2\left(-\dfrac{1}{4}\sqrt{\beta^2\gamma^2} - \dfrac{\beta\gamma}{2}\right)}{\beta^2} - \frac{3\gamma\left[\sec\left(x - \dfrac{\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right) \pm \tan\left(x - \dfrac{\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right)\right]^2}{2\beta}.$$

Assign $P = \dfrac{r^2}{4}$, $Q = \dfrac{r^2 - 2}{2}$, $R = \dfrac{r^2}{4}$, from Table 1 this assignment corresponds to $F = \mathrm{sn} \mp i\mathrm{cn}$, so the solution can be found as $r \to 1$

$$u_{15,16}(x,t) = \frac{2\left(\dfrac{\beta\gamma}{2} - \dfrac{1}{4}\sqrt{\beta^2\gamma^2}\right)}{\beta^2} - \frac{3\gamma\left[\tanh\left(x - \dfrac{\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right) \pm i\,\mathrm{sech}\left(x - \dfrac{\sqrt[4]{\beta^2\gamma^2}t^\alpha}{\alpha\sqrt{\beta}}\right)\right]^2}{2\beta}.$$

With $P = \dfrac{1}{4}, Q = \dfrac{1 - 2r^2}{2}, R = \dfrac{1}{4}$, from Table 1 this choice follows $F = \dfrac{\mathrm{sn}}{1 \pm \mathrm{cn}}$ so that the solution can be constructed as for $r \to 0$

$$u_{17,18}(x,t) = \frac{2\left(-\frac{1}{4}\sqrt{\beta^2\gamma^2} - \frac{\beta\gamma}{2}\right)}{\beta^2} - \frac{3\gamma \sin^2\left(x - \frac{\sqrt[4]{\beta^2\gamma^2}t^{\alpha}}{\alpha\sqrt{\beta}}\right)}{2\beta\left[1 \pm \cos\left(x - \frac{\sqrt[4]{\beta^2\gamma^2}t^{\alpha}}{\alpha\sqrt{\beta}}\right)\right]^2}.$$

Suppose $P = \dfrac{r^2}{4}$, $Q = \dfrac{r^2-2}{2}$, $R = \dfrac{1}{4}$, from Table 1 this choices correspond to $F = \dfrac{\mathrm{sn}}{1 \pm \mathrm{dn}}$, so if $r \to 1$ solution can be stated as:

$$u_{19,20}(x,t) = \frac{2\left(\frac{\beta\gamma}{2} - \frac{1}{4}\sqrt{\beta^2\gamma^2}\right)}{\beta^2} - \frac{3\gamma \tanh^2\left(x - \frac{\sqrt[4]{\beta^2\gamma^2}t^{\alpha}}{\alpha\sqrt{\beta}}\right)}{2\beta\left[1 \pm \mathrm{sech}\left(x - \frac{\sqrt[4]{\beta^2\gamma^2}t^{\alpha}}{\alpha\sqrt{\beta}}\right)\right]^2}.$$

While $P = \dfrac{1-r^2}{4}$, $Q = \dfrac{r^2+1}{2}$, $R = \dfrac{1-r^2}{4}$, it can be deducted from Table 1 $F = \dfrac{\mathrm{cn}}{1 \mp \mathrm{sn}}$, then the solution can be evaluated while $r \to 0$

$$u_{21,22}(x,t) = \frac{2\left(-\frac{1}{4}\sqrt{\beta^2\gamma^2} - \frac{\beta\gamma}{2}\right)}{\beta^2} - \frac{3\gamma \cos^2\left(x - \frac{\sqrt[4]{\beta^2\gamma^2}t^{\alpha}}{\alpha\sqrt{\beta}}\right)}{2\beta\left[1 \pm \sin\left(x - \frac{\sqrt[4]{\beta^2\gamma^2}t^{\alpha}}{\alpha\sqrt{\beta}}\right)\right]^2}.$$

When P, Q, R are set as $P = \dfrac{(1-r^2)^2}{4}$, $Q = \dfrac{r^2+1}{2}$, $R = \dfrac{1}{4}$, F is going to be as $F = \dfrac{\mathrm{sn}}{\mathrm{dn} \mp \mathrm{cn}}$ from Table 1, so the solution can be obtained as for $r \to 0$

$$u_{23,24}(x,t) = \frac{2\left(-\frac{1}{4}\sqrt{\beta^2\gamma^2} - \frac{\beta\gamma}{2}\right)}{\beta^2} - \frac{3\gamma \sin^2\left(x - \frac{\sqrt[4]{\beta^2\gamma^2}t^{\alpha}}{\alpha\sqrt{\beta}}\right)}{2\beta\left[1 \pm \cos\left(x - \frac{\sqrt[4]{\beta^2\gamma^2}t^{\alpha}}{\alpha\sqrt{\beta}}\right)\right]^2}.$$

5 Wave behaviors and graphical simulations of solutions

In this part we give the 3D and 2D illustrations of some trigonometric function solutions and solitonic solutions of the considered equation, in Fig. 1 the soliton solution for $u_1(x,t)$ respectively in 3D and 2D. Also graphical illustrations of the travelling wave solution for $u_{11}(x,t)$ are given in Fig. 2. In Fig. 3 and Fig. 4, the graphical simulations of the periodic wave solutions for $u_{13}(x,t)$ and $u_{23}(x,t)$ are given respectively.

6 Conclusion

In this article, Jacobi elliptic function expansion method is employed to construct the wave solutions of time fractional Benjamin–Ono equation which corresponds to the solitary internal waves in deep water. The results show that Jacobi elliptic function expansion method is efficient, reliable, applicable and used as clear, simple algorithm in pro-

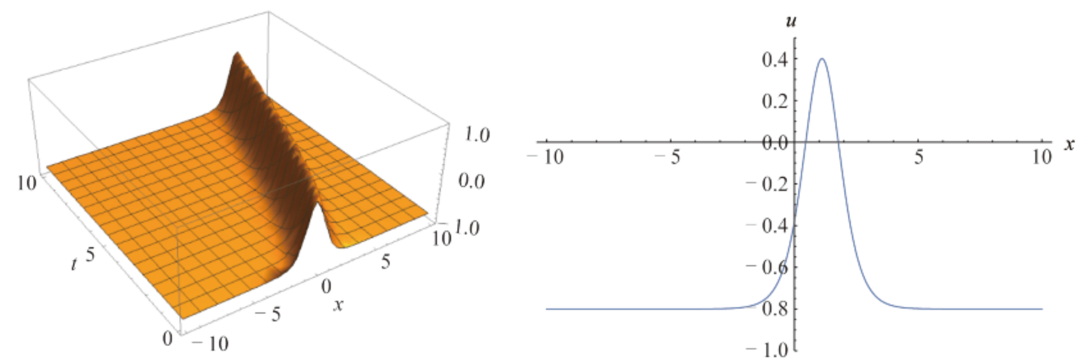

Fig. 1. 3D and 2D graphics of the solitary wave solution $u_1(x,t)$ for $\beta = 1$, $\gamma = 0.2$.

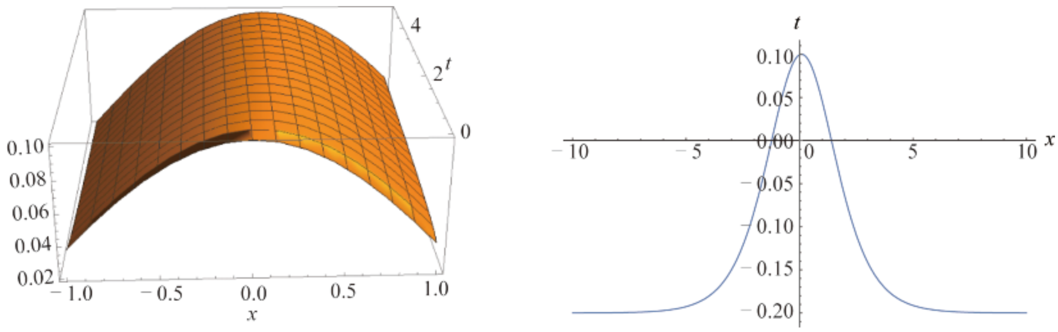

Fig. 2. 3D and 2D graphical illustrations of the wave solution $u_{11}(x,t)$ for $\beta = 0.005,\ \gamma = 0.001$.

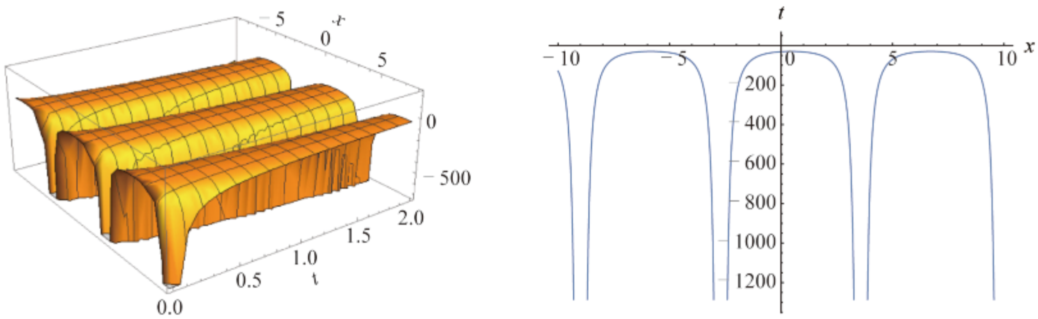

Fig. 3. 3D and 2D graphical representation of traveling wave solution $u_{13}(x,t)$ for $\beta = 0.05,\ \gamma = 1$.

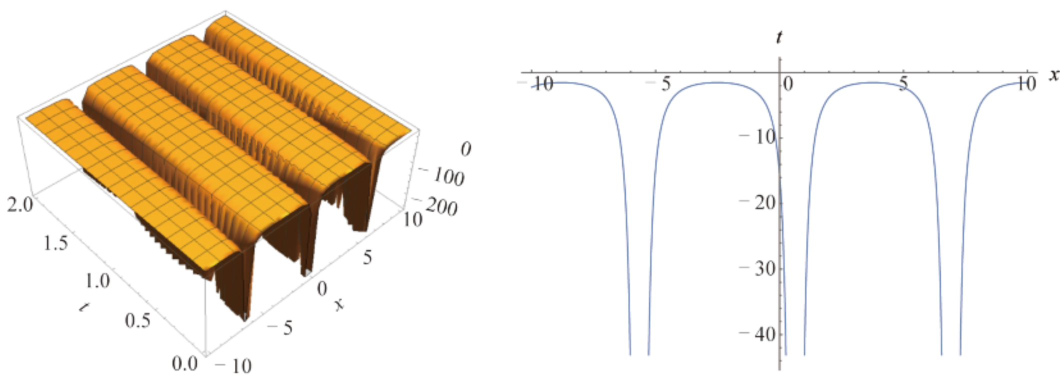

Fig. 4. 3D and 2D graphical simulations of the wave solution $u_{23}(x,t)$ for $\beta = 0.1,\ \gamma = 0.1$.

gramming. At the same time some 2D and 3D graphical illustrations of some solutions are given. Clearly it is understood that Jacobi elliptic expansion method is a good tool to obtain the exact solutions of fractional partial differential equations where the derivatives are in conformable sense.

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