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Yücel Çenesiz, Ali Kurt, Erkan Nane

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STOCHASTIC SOLUTIONS OF CONFORMABLE FRACTIONAL CAUCHY PROBLEMS

YÜCEL ÇENESİZ, ALI KURT, AND ERKAN NANE

ABSTRACT. In this paper we give stochastic solutions of conformable fractional Cauchy problems. The stochastic solutions are obtained by running the processes corresponding to Cauchy problems with nonlinear deterministic clocks.

1. INTRODUCTION

Scientists have paid great attention to fractional calculus which is known as the differentiation and integration of arbitrary order. In the last few decades, great amount of work was carried out on fractional calculus in different fields of engineering and science extensively with variety of applications. While these studies have been carried out, scientists used different definitions of fractional derivative and integral such as Grünwald-Lethnikov, Riesz-Fischer, Dzherbashyan-Caputo, Riemann-Liouville, modified Riemann-Liouville and etc. But almost all of these derivatives have some kind of flaws. For instance, the Riemann-Liouville fractional derivative of a constant is not zero, the Riemann-Liouville derivative and Dzherbashyan-Caputo derivative do not obey the Leibnitz rule and chain rule. The Riemann-Liouville derivative and Dzherbashyan-Caputo do not satisfy the known formula of the derivative of the quotient of two functions. The Dzherbashyan-Caputo definition assumes that the function f is smooth (at least absolutely continuous). Recently a new type of fractional derivative called conformable fractional derivative that overcomes these flaws has been introduced by R. Khalil et al. [12].

Definition 1.1. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a function. The α^{th} order "conformable fractional derivative" of f is defined by,

$$T_\alpha(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > 0, \alpha \in (0, 1)$. If f is α -differentiable in some $(0, a), a > 0$ and $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ and the "conformable fractional integral" of

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a function f starting from $a \geq 0$ is defined as:

$$I_{\alpha}^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$.

The following theorem gives some properties which are satisfied by the conformable fractional derivative.

Theorem 1.2 ([12]). *Let $\alpha \in (0, 1]$ and suppose f, g are α -differentiable at point $t > 0$. Then*

- (1) $T_{\alpha}(cf + dg) = cT_{\alpha}(f) + dT_{\alpha}(g)$ for all $a, b \in \mathbb{R}$.
- (2) $T_{\alpha}(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.
- (3) $T_{\alpha}(\lambda) = 0$ for all constant functions $f(t) = \lambda$.
- (4) $T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f)$.
- (5) $T_{\alpha}\left(\frac{f}{g}\right) = \frac{gT_{\alpha}(f) - fT_{\alpha}(g)}{g^2}$.
- (6) In addition, if f differentiable, then $T_{\alpha}(f)(t) = t^{1-\alpha} \frac{df}{dt}$.

Because of its applicability, and effectiveness scientists studied conformable derivative in various fields. For instance; Abdeljawad [1] has presented basic principles of calculus such as chain rule, integration by parts and etc.; Ghanbari and Ghoulami [9] used the so called fractional conformable operators for establishing fractional Hamiltonian systems; Iyiola and Nwaeze [10] proved some results on conformable fractional derivatives and fractional integrals, also they apply the D’Alambert approach to a conformable fractional differential equation; Anderson [2] employed the conformable derivative to formulate several boundary value problems with three or four conformable derivatives, including those with conjugate, right-focal, and Lidstone conditions; Zhao *et al.* [22] investigate a new concept of conformable delta fractional derivative which has the identity operator on time scales; Pospisil and Skripkova [19] proved Sturm’s separation and Sturm’s comparison theorems for differential equations involving a conformable fractional derivative of order $0 < \alpha \leq 1$ using local properties of conformable fractional derivative. Hence, it is worthwhile to work on this new area.

In this paper, we will study the stochastic solution of equations with conformable time derivative where the space operators may correspond to fractional Brownian motion, or a Levy process, or a general semigroup in a Banach space, or a process killed upon exiting a bounded domain in $\mathbb{R}^d$.

Self-similar processes arise naturally in limit theorems of random walks and other stochastic processes, and they have been applied to model various phenomena in a wide range of scientific areas including physics, finance, telecommunications, turbulence, image processing, hydrology and economics. The most important example of self-similar processes is fractional Brownian motion (fBm) which is a stationary and

centered Gaussian process $B^H = \{B^H(t), t \geq 0\}$ with $B^H(0) = 0$ and covariance function

$$(1.1) \quad E[B^H(s)B^H(t)] = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s - t|^{2H}).$$

where $H \in (0, 1)$ is a constant. By using (1.1) one can verify that B^H is self-similar with index H (i.e., for all constants $c > 0$, the processes $\{B^H(ct), t \geq 0\}$ and $\{c^H B^H(t), t \geq 0\}$ have the same finite-dimensional distributions) and has stationary increments. When $H = 1/2$, B^H is a Brownian motion, which will be written as B . This process was studied and popularized in one-dimension by Mandelbrot and Van Ness [13].

One of the main results in this paper is the following theorem.

Theorem 1.3. *Let $\alpha = 2H \in (0, 1)$, and $B^H(t)$ be fractional Brownian motion of index $H \in (0, 1/2)$ started at 0. Then the unique solution of the equation with conformable time derivative*

$$(1.2) \quad \begin{aligned} T_{2H}u(t, x) &= H\Delta_x u(t, x), t > 0, x \in \mathbb{R}^d \\ u(0, x) &= f(x), x \in \mathbb{R}^d \end{aligned}$$

is given by

$$(1.3) \quad u(t, x) = E[f(x + B^H(t))]$$

Remark 1.4. O'Malley *et al.* [17] studied a multiple dimensional extension of fBm. O'Malley and Cushman [18] developed a nonlinear extension of classical fBm, which is fBm run with a nonlinear clock. Garra *et al.* [8] also considered solutions of equations like (1.2) and its generalizations, and obtained some analytic expressions for the solutions using Wright functions. In the last theorem and the other results in this paper we observe that replacing the first time derivative with a conformable time fractional derivative corresponds to replacing the time in the stochastic solution of a Cauchy problem with a nonlinear clock. Please see Meerschaert and Sheffler [16] for the corresponding time change when we work with a time fractional Cauchy problem with a Dzherbashyan-Caputo fractional derivative.

Our paper is organized as follows. In the next section we give some preliminaries about semigroups on a Banach space and Lévy semigroups. In section 3, we state main theorems and give the proof of Theorem 1.3.

2. PRELIMINARIES ON LÉVY PROCESSES

Given a Banach space and a bounded continuous semigroup $P(t)$ on that space with generator L_x , it is well known that $u(t, x) = P(t)f(x)$ is the unique solution to the abstract Cauchy problem

$$(2.1) \quad \frac{\partial}{\partial t}u(t, x) = L_x u(t, x); \quad u(0, x) = f(x)$$

for any f in the domain of L_x ; see, for example, [4, 20].

In the case where $X_0(t)$ is a Lévy process started at zero and $X(t) = x + X_0(t)$ for $x \in \mathbb{R}^d$, the generator L_x of the semigroup $P(t)f(x) = E_x[f(X(t))]$ is a pseudo-differential operator [3, 11, 21] that can be explicitly computed by inverting the Lévy representation. The Lévy process $X_0(t)$ has characteristic function

$$E[\exp(ik \cdot X_0(t))] = \exp(t\psi(k))$$

with

$$\psi(k) = ik \cdot a - \frac{1}{2}k \cdot Qk + \int_{y \neq 0} \left(e^{ik \cdot y} - 1 - \frac{ik \cdot y}{1 + \|y\|^2} \right) \phi(dy),$$

where $a \in \mathbb{R}^d$, Q is a nonnegative definite matrix, and ϕ is a σ -finite Borel measure on $\mathbb{R}^d$ such that

$$\int_{y \neq 0} \min\{1, \|y\|^2\} \phi(dy) < \infty;$$

see, for example, [14, Theorem 3.1.11] and [3, Theorem 1.2.14]. Let

$$\hat{f}(k) = \int_{\mathbb{R}^d} e^{-ik \cdot x} f(x) dx$$

denote the Fourier transform.

Define the set functions

$$D(L_x) = \{f \in L^1(\mathbb{R}^d) : \psi(k)\hat{f}(k) = \hat{h}(k) \exists h \in L^1(\mathbb{R}^d)\}.$$

Theorem 3.1 in [5] shows that $L_x f(x)$ is the inverse Fourier transform of $\psi(k)\hat{f}(k)$ for all $f \in D(L_x)$, the domain of L_x , and L_x is given by

$$(2.2) \quad \begin{aligned} L_x f(x) &= a \cdot \nabla f(x) + \frac{1}{2} \nabla \cdot Q \nabla f(x) \\ &+ \int_{y \neq 0} \left(f(x+y) - f(x) - \frac{\nabla f(x) \cdot y}{1 + y^2} \right) \phi(dy) \end{aligned}$$

for all $f \in W^{2,1}(\mathbb{R}^d)$, the Sobolev space of L^1 -functions whose first and second partial derivatives are all L^1 -functions.

We can also view the infinitesimal generator of a Lévy process as $L_x = \psi(-i\nabla)$ where $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_d)$. Some examples are as follows. When $\psi(k) = -D\|k\|^\alpha$, then $L_x = -D(-\Delta)^{\alpha/2}$, a fractional derivative in space, using the correspondence $k_j \rightarrow -i\partial/\partial x_j$ for $1 \leq j \leq d$. The corresponding Lévy process $X_0(t)$ is spherically symmetric stable α -stable process. When $\psi(k) = D \sum_j (ik_j)^{\alpha_j}$, $L_x = D \sum_j \partial^{\alpha_j} / \partial x_j^{\alpha_j}$ using Riemann-Liouville fractional derivatives in each variable. The corresponding Lévy process X_0 has independent stable marginals. This form is different from fractional Laplacian unless all $\alpha_j = 2$.

3. MAIN RESULTS

Our first main result is the following theorem.

Theorem 3.1. *Let $\alpha \in (0, 1)$, and L_x be the generator of a continuous (Markov) semigroup $P(t)f(x) (= E_x[f(X_t)])$, and take $f \in D(L_x)$, the domain of the generator. Then, the unique solution of the equation with conformable time derivative*

$$(3.1) \quad \begin{aligned} T_\alpha u(t, x) &= L_x u(t, x), t > 0 \\ u(0, x) &= f(x), \end{aligned}$$

is given by

$$(3.2) \quad u(t, x) = P(t^\alpha/\alpha)f(x) (= E_x[f(X_{t^\alpha/\alpha})])$$

Proof. The proof follows by a chain rule when the derivative of the semigroup, $t \rightarrow P(t)$ exists in the usual sense. Let $u(t, x) = P(t^\alpha/\alpha)f(x)$. Then by chain rule

$$(3.3) \quad \begin{aligned} \frac{\partial}{\partial t} u(t, x) &= P'(t^\alpha/\alpha)f(x) \frac{dt^\alpha/\alpha}{dt} \\ &= t^{\alpha-1} P'(t^\alpha/\alpha)f(x) \\ &= t^{\alpha-1} L_x P(t^\alpha/\alpha)f(x) \\ &= t^{\alpha-1} L_x u(t, x). \end{aligned}$$

Hence we have

$$T_\alpha u(t, x) = t^{1-\alpha} \frac{\partial}{\partial t} u(t, x) = L_x u(t, x).$$

Uniqueness holds since the solution of the the equation (2.1) is unique and the solution of the conformable Cauchy problem is given in terms of the solution of equation (2.1). □

Proof of Theorem 1.3. This follows from the fact that the density of fractional Brownian motion, $p(t, x, y) = \frac{1}{(2\pi t^{2H})^{d/2}} \exp(-|x - y|^2/2t^{2H})$ satisfy

$$\frac{\partial}{\partial t} p(t, x, y) = H t^{2H-1} \Delta_x p(t, x, y)$$

Hence we get

$$T_{2H} p(t, x, y) = H \Delta_x p(t, x, y)$$

Next we can apply the conformable fractional derivative and use the dominated convergence theorem to the function

$$u(t, x) = E[f(x + B^H(t))] = \int_{\mathbb{R}^d} p(t, x, y) f(y) dy$$

to get

$$\begin{aligned}
 (3.4) \quad T_{2H}u(t, x) &= \int_{\mathbb{R}^d} T_{2H}p(t, x, y)f(y)dy \\
 &= \int_{\mathbb{R}^d} H\Delta_x p(t, x, y)f(y)dy \\
 &= H\Delta_x \int_{\mathbb{R}^d} p(t, x, y)f(y)dy \\
 &= H\Delta_x u(t, x).
 \end{aligned}$$

□

The next result is a restatement of Theorem 3.1 for Lévy semigroups. **The proof does not use the differentiability of the semigroup $T(t)$** , rather it relies on a Fourier transform argument. Recall the following notation for the Fourier transform:

$$\hat{u}(t, k) = \int_{\mathbb{R}^d} e^{-ik \cdot x} u(t, x) dx;$$

Theorem 3.2. Suppose that $X(t) = x + X_0(t)$ where $X_0(t)$ is a Lévy process starting at zero. If L_x is the generator (2.2) of the semigroup $P(t)f(x) = E_x[(f(X_t))]$ on $L^1(\mathbb{R}^d)$, then, for any $f \in D(L_x)$, the unique solution of the following Cauchy problem

$$\begin{aligned}
 (3.5) \quad T_\alpha u(t, x) &= L_x u(t, x), t > 0 \\
 u(0, x) &= f(x),
 \end{aligned}$$

is given by

$$(3.6) \quad u(t, x) = P(t^\alpha/\alpha)f(x) (= E_x[f(X_{t^\alpha/\alpha}])$$

Proof. Take Fourier transforms on both sides of (3.5) to get

$$(3.7) \quad T_\alpha \hat{u}(t, k) = \psi(k) \hat{u}(t, k)$$

using the fact that $\psi(k)\hat{f}(k)$ is the Fourier transform of $L_x f(x)$. Then since the eigenfunctions of the conformable derivative operator

$$T_\alpha g(t) = \lambda g(t); f(0) = c$$

are given by $g(t) = ce^{\frac{\Delta}{\alpha}t^\alpha}$, the solution of equation (3.7) are given by

$$\hat{u}(t, k) = ce^{\frac{\psi(k)}{\alpha}t^\alpha} = \hat{f}(k)e^{\frac{\psi(k)}{\alpha}t^\alpha} = \hat{f}(\text{Characteristic function of } X_{t^\alpha/\alpha})$$

since the initial function is $f(x)$ we have $\hat{u}(0, k) = \hat{f}(k)$, and the Fourier transform (characteristic function) of the Lévy process X_t is $e^{t\psi(k)}$, we have by taking inverse Fourier transforms (this is given by the convolution of f and the density of $X_{t^\alpha/\alpha}$)

$$u(t, x) = \int_{\mathbb{R}^d} p(t^\alpha/\alpha, x - y)f(y)dy = E_x[f(X_{t^\alpha/\alpha})]$$

□

Let $D \subset \mathbb{R}^d$ be a bounded domain with a nice (smooth) boundary ∂D . Let $p_D(t, x, y)$ denote the heat kernel of the equation

$$(3.8) \quad \begin{aligned} \frac{\partial}{\partial t} u(t, x) &= L_x^D u(t, x), \quad x \in D, \quad t > 0 \\ u(0, x) &= f(x), \quad x \in D \\ u(t, x) &= 0, \quad x \in \partial D, \quad \text{when } \beta = 2 \\ u(t, x) &= 0, \quad x \in D^C, \quad \text{when } 0 < \beta < 2 \end{aligned}$$

here L_x^D is the generator of the stable Lévy process of index $\beta \in (0, 2]$ killed at the first exit time, τ_D , from the domain D . When $\beta = 2$, X_t is Brownian motion and $L_x^D = \Delta_x|_D$ is the Laplacian restricted to the bounded domain D . When $\beta \in (0, 2)$, then X_t is a stable Lévy process and $L_x^D = -(\Delta)^{\beta/2}|_D$ is the fractional Laplacian restricted to the bounded domain D .

There exist eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots$, such that $\lambda_n \rightarrow \infty$, as $n \rightarrow \infty$, with the corresponding complete orthonormal set (in H_0^2) of eigenfunctions ϕ_n of the operator L_D satisfying

$$(3.9) \quad L_x^D \phi_n(x) = -\lambda_n \phi_n(x), \quad x \in D; \phi_n|_{\partial D} = 0 \quad \text{when } \beta = 2; \phi_n|_{D^C} = 0 \quad \text{when } 0 < \beta < 2.$$

A well known fact is that

$$(3.10) \quad p_D(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y) \quad \text{for all } x, y \in D, t > 0.$$

The series converges absolutely and uniformly on $[t_0, \infty) \times D \times D$ for all $t_0 > 0$.

The unique solution of equation (3.8) is given by

$$u(t, x) = \int_D p_D(t, x, y) f(y) dy = E_x[f(X_t) I(\tau_D > t)] = \sum_{n=1}^{\infty} \phi_n(x) e^{-\lambda_n t} \left[\int_D f(y) \phi_n(y) dy \right],$$

where here X_t is a Lévy process and $\tau_D = \inf\{s > 0 : X(s) \notin D\}$ is the first exit time of the process from D . See, for example, [6, 15] for more on the killed stable process and its generator and semigroup.

Now we can state our theorem for the conformable Cauchy problem in bounded domains.

Theorem 3.3. Suppose that $X(t) = x + X_0(t)$ where $X_0(t)$ is a stable Lévy process of index $\beta \in (0, 2]$ starting at zero. If L_x^D is the generator (2.2) of the semigroup $P^D(t)f(x) = E_x[f(X_t)I(\tau_D > t)]$ on $H_0^2(D)$, then, for any $f \in D(L_x^D)$, the unique

solution of the following initial-boundary value problem

$$\begin{aligned}
 (3.11) \quad & T_\alpha u(t, x) = L_x^D u(t, x), \quad t > 0, \quad x \in D \\
 & u(0, x) = f(x) \\
 & u(t, x) = 0, \quad x \in \partial D, \quad \text{when } \beta = 2 \\
 & u(t, x) = 0, \quad x \in D^C, \quad \text{when } 0 < \beta < 2
 \end{aligned}$$

is given by

$$\begin{aligned}
 (3.12) \quad & u(t, x) = T^D(t^\alpha/\alpha)f(x)(= E_x[f(X_{t^\alpha/\alpha})I(\tau_D > t^\alpha/\alpha)]) \\
 & = \sum_{n=1}^{\infty} \phi_n(x) e^{-\lambda_n t^\alpha/\alpha} \left[\int_D f(y) \phi_n(y) dy \right]
 \end{aligned}$$

Proof. The proof follows by a separation of variables as in [7, section 2] and as in the proof of Theorem 5.1 in [6]. We leave the details to the reader. $\square$

When $D = [0, L] \subset \mathbb{R}$ and $\beta = 2$, then $\lambda_n = (n\pi/L)^2$, $\phi_n(x) = \sin(n\pi x/L)$. In this case we recover the result in Çenesiz and Kurt [7, section 2].

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YÜCEL ÇENESİZ, DEPARTMENT OF MATHEMATICS, SELCUK UNIVERSITY, KONYA, TURKEY

ALI KURT, DEPARTMENT OF MATHEMATICS, MUSTAFA KEMAL UNIVERSITY, HATAY , TURKEY

ERKAN NANE, DEPARTMENT OF MATHEMATICS AND STATISTICS, AUBURN UNIVERSITY, AUBURN, AL 36849

E-mail address: ezn0001@auburn.edu